People's Democratic Republic of Algeria

Ministry of Higher Education and Scientific Research

Ibn Khaldoun University - Tiaret –

FACULTY OF MATERIAL SCIENCES

Department of Physics

COURSE AND TUTORIAL HANDOUTS

Heat Transfer1

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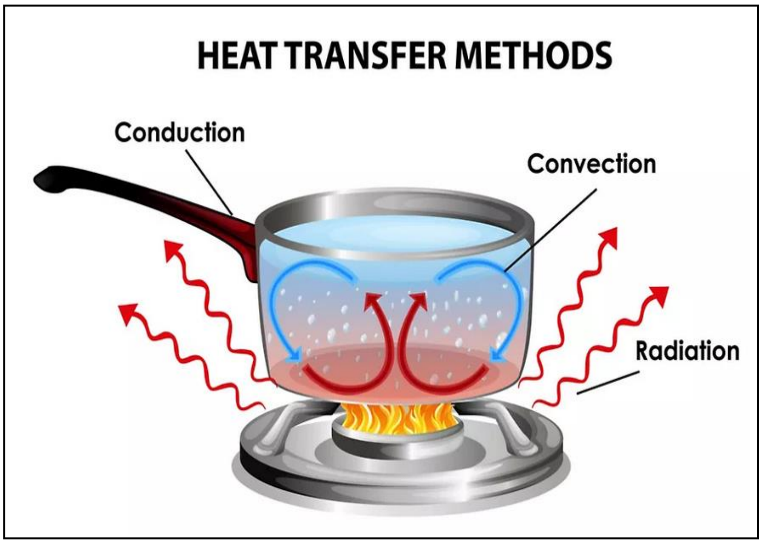

ACADEMIC YEAR : 2024-2025

DEDICACTION

Heat transfer is a fundamental discipline that lies at the heart of our environment and our technologies. Whether it's regulating the temperature of the human body, optimizing the performance of an engine, designing air conditioning systems or improving the energy efficiency of buildings, the principles of heat transfer are constantly at work. Mastering heat transfer is therefore essential for engineers and scientists, enabling them to develop innovative solutions, improve the efficiency of existing systems and meet the energy challenges of tomorrow, while ensuring comfort, safety and performance.

This handout is the product of four years of experiences teaching heat transfer in the Physics Department at the University of Tiaret. It's primarily aimed at Material Science students (3rd year LMD Energy Physics and Fundamental Physics), but also at any other higher education student looking to deepen their knowledge of heat transfer.

The objective of this manuscript is to provide a comprehensive overview of heat transfer, particularly the phenomenon of conduction. It starts with the fundamentals of heat transfer (Chapter 1), then it progresses to the study of steady-state conduction, exploring one-dimensional conduction in Chapter 2 and two-dimensional conduction in Chapter 3, using both analytical and numerical tools (finite difference method). Chapter 4 is dedicated to solve the transient heat equation, relying on various techniques such as the Laplace transform, graphical and tabular methods, and the method of variable separation.

To reinforce skill acquisition, solved exercises are included at the end of each chapter. We sincerely hope this handout serves as a practical and relevant reference for everyone interested in the study of heat transfer.

NOMENCLATURES

A Surface [m^2]

q Heat flux [W/(m^2)]

λ Thermal conductivity [W/(m.K)]

$\frac{\partial T}{\partial n}$ Temperature gradient [K/m]

$\dot{E}_g$ Rate of heat generation [W]

dx Elementary position in x direction [m]

dy Elementary position in y direction [m]

dz Elementary position in z direction [m]

$\dot{q}$ Rate of heat generation per unit volume [W/m^3]

$\dot{E}_{st}$ Rate of energy stored in the control volume [W]

$\dot{E}_{in}$ Energy inflow [W]

$\dot{E}_{out}$ Energy outflow [W/m^3]

q_{conv} Convective heat flux [W/m^2]

h Convection heat transfer coefficient [W/(m^2.K)]

T_∞ Fluid temperature[K]

T Temperature [K]

ρ Density [kg/(m^3)]

C_p Specific heat [J/(kg.K)]

t Time [s]

r Radius [m]

ϕ Rtae of heat transfer [W]

σ Stefan Boltzman constant [W/(m$^{2}\cdot$K^4)]

L_c Characteristic length [m]

R_{th} Thermal resistance [°C/W]

F_o Fourier number

Bi Biot number

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CHAPTER 1: INTRODUCTION TO HEAT TRANSFER

1.1 INTRODUCTION

As known from experience, thermodynamics focuses on systems in equilibrium and the transitions between their states. In contrast, heat transfer deals with systems that are not in thermal equilibrium, classifying it as a non-equilibrium process. Consequently, a complete understanding of heat transfer requires more than just the principles of thermodynamics. Nevertheless, the laws of thermodynamics provide the fundamental principles for studying heat transfer. The first law states that the rate of energy input to a system equals to the rate of energy stored in it. The second law of thermodynamics defines the natural direction of heat flow, stating that heat spontaneously transfers from the high-temperature region to the low-temperature regions.

A fundamental requirement for heat transfer is the presence of a temperature difference. Without this difference between two mediums, heat transfer cannot occur. This temperature difference drives heat transfer in much the same way that a voltage difference causes electric current to flow, or a pressure difference induces fluid motion. The rate of heat transfer in a specific direction is directly proportional to the temperature gradient defined as the temperature difference per unit length. Consequently, A greater temperature gradient corresponds to a higher heat transfer rate.

1.2 DEFINITIONS

<u>**System**</u>:

A system, as shown in figure 1.1, refers to a specific portion of the universe under investigation, all the rest being considered as the surroundings. A closed system is characterized by the absence of mass transfer across its boundaries. Conversely, a control volume, as illustrated in figure 1.2, is a defined region in space that allows both mass and energy to cross its boundaries.

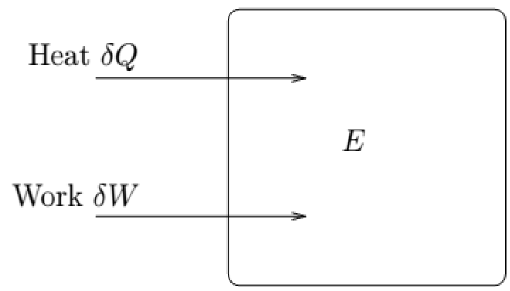


Figure 1.1: System with work and heat flow .

The thermodynamic state of a system is defined by a specific set of property values required to fully describe it. A system is in equilibrium when its state remains constant over time. The state of matter can exist in different phases: solid, liquid, or gas.

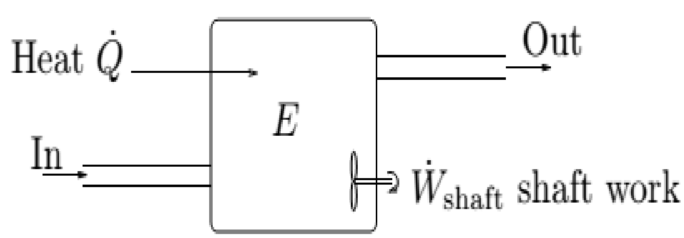


Figure 1.2: Control volume with flow in and out, heat flow in, and shaft work out .

<u>Steady state</u>

Steady state describes a state where the temperature profile within a material remains constant over time. In this state, all heat introduced or produced within a substance is simultaneously transferred away. This concept is fundamental to engineering applications such as choosing insulation for superheated steam pipes to prevent heat loss, developing extended surfaces (like fins) to efficiently cool electronic devices or air-cooled engines, and maintaining stable operation in nuclear systems .

Unsteady state:

When the rate of internal heat generation or the boundary conditions of an object change, its temperature distribution also changes over time. This state is called as unsteady state or transient. Processes such as heating and cooling exemplify this behaviour. Mathematically, these changes force the temperature distribution to evolve .

1.3 MODES OF HEAT TRANSFER

There are three modes of heat transfer: conduction, convection, and radiation. In this section, these modes of heat transfer are summarized.

1.3.1 Conduction Heat Transfer

Conduction heat transfer happens across in all three states of matter: solids, liquids, and gases. While often associated with solids in engineering applications, this phenomenon is also a significant mechanism in fluids, even when fluid motion is present. Fundamentally, conduction phenomenon is the transfer of thermal energy from higher to lower temperatures within a medium through molecular diffusion, without any bulk movement of the medium itself. (Figure. 1.3.a).

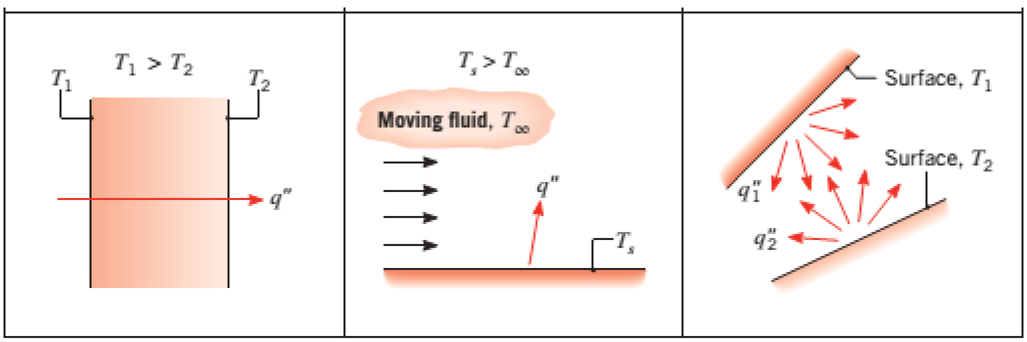
The term diffusion is also used to express conduction. Diffusion heat transfer or conduction heat transfer are two statements that are equivalent in many heat transfer reports and papers.

1.3.2 Convection Heat Transfer

Convection heat transfer is a phenomenon observed in both liquid and gaseous phases. It is characterized by the bulk movement of the fluid, resulting in the transfer of thermal energy from one point to another within the fluid domain.

Convection heat transfer is directly dependent on fluid motion. If a fluid is motionless, convection does not occur. A typical example is the transfer of heat from a heated plate to a cooler fluid flowing around it. The moving fluid carries thermal energy away from the plate surface and distributes it throughout the fluid domain (Figure 1.3.b).

The principal law describing this phenomenon is Newton's law of Cooling. In practical terms, there are three distinct types of convective heat transfer.



(a) (b) (c)

Figure1.3: Conduction, convection, and radiation heat transfer modes.

<u>Forced convection</u>: this phenomenon occurs when an external force, like a fan or pump, causes the fluid to move and heat to transfer.

<u>Natural convection</u> is a heat transfer mechanism in which fluid motion is driven by density gradients within the fluid under the influence of gravity (or other body forces). According to Archimedes' principle, less dense regions undergo an upward buoyant force, causing them to move. This phenomenon is also known in the literature as free convection.

Mixed convection is essentially forced convection where natural convection also plays a significant role and can't be ignored. For this reason, the buoyancy force must be taken in the equations that describe the flow.

1.3.3 Radiation Heat Transfer

Thermal radiation is a fundamental heat transfer mechanism where all matter with a temperature above absolute zero emits energy as electromagnetic waves (photons). Unlike conduction and convection, radiation does not require a material medium and can, therefore, propagate through a vacuum. As illustrated in (Figure 1.3 c), a net heat transfer occurs between surfaces at different temperatures, with energy flowing from the hotter to the colder body.

1.4 CONDUCTION HEAT TRANSFER PROCESS

Energy transfer by conduction is accomplished in two ways:

The first mechanism: Energy transfer through conduction occurs primarily through molecular interaction and is the most universal mode of heat transfer. When a substance is heated, its molecules gain kinetic energy and vibrate more intensely. These highly energetic molecules then collide with adjacent, less energetic molecules, transferring some of this vibrational energy.

This process continues, causing a chain reaction of energy transfer throughout the material. This mechanism is present in all states of matter solids, liquids, and gases whenever a temperature gradient exists.

The second mechanism of conduction heat transfer is through free electrons. In materials like pure metals, a high number of unbound electrons are able to move throughout the atomic lattice. These electrons are highly efficient carriers of thermal energy, significantly contributing to the overall heat transfer. This explains why pure metals are excellent thermal conductors: their high concentration of free electrons allows for rapid energy transport. This characteristic is greatly reduced in alloys and is almost absent in non-metallic solids.

1.5 FOURIER'S LAW

Fourier's Law is the fundamental principle of conduction heat transfer. It states that the heat flux (rate of heat transfer per unit area) is directly proportional to the temperature gradient in a specific direction. The constant of proportionality in this relationship is the thermal conductivity λ, which is a material property that can vary with temperature. The rate of heat transfer by conduction is determined as follows:

$$\phi = -\lambda \left\| \overrightarrow{gradT} \right\| . A = -\lambda \frac{\partial T}{\partial n} A \tag{1.1}$$

In many systems, the area A is not constant but varies with the distance along the n direction.

The temperature gradient is a vector oriented normal to the isothermal surface, and it corresponds mathematically to the directional derivative of the temperature field in that direction.

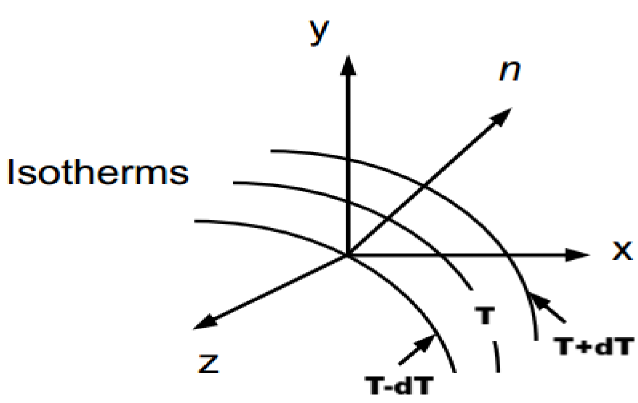


Figure. 1.4: Temperature vectors

1.6 THERMOPHYSICAL PROPERTIES

Understanding thermophysical properties is essential for analyzing and predicting conduction heat transfer. These properties describe the manner in which a material responds to and participates in the transfer of thermal energy. For conduction, where heat is transferred through a stationary medium by molecular vibration and free electron movement, three properties are particularly important: thermal conductivity, specific heat capacity, and density, which combine to form thermal diffusivity.

Thermal Conductivity

Thermal conductivity (λ) quantifies a material's ability to conduct heat. Simply put, it's a measure of how easily heat can flow through a substance, its unit is [W/m.K]. Materials with high thermal conductivity, like metals (e.g., copper, aluminium), are excellent heat conductors and feel "cold" to the touch because they quickly draw heat away from your hand. Conversely, materials with low thermal conductivity, such as insulation, wood, or air, are poor conductors and are thus good insulators.

The value of λ depends significantly on the material's microstructure, temperature, and pressure. For example, in solids, heat is conducted by lattice vibrations (phonons) and free electrons. Metals have a high concentration of free electrons, which makes them excellent conductors. In liquids and gases, molecular collisions are the primary mechanism for heat transfer, leading to generally much lower thermal conductivities compared to solids .

Specific Heat Capacity :

The specific heat capacity (Cp): represents the amount of energy required to raise the temperature of a unit mass of a substance by one degree Celsius (or Kelvin). It is generally expressed in [J/kg.K]. It essentially tells you how much thermal energy a material can store. Water, for instance, has a very high specific heat capacity, meaning it can absorb a lot of heat without a significant temperature change, which is why it's used extensively in cooling systems. Materials with low specific heat capacity heat up and cool down quickly.

Like thermal conductivity, the specific heat capacity varies with temperature and phase. For most engineering applications, specific heat is often treated as constant over a small temperature range, but for larger variations, its temperature dependence must be considered.

Density

Density (ρ) is defined as the mass per unit volume of a material, so its unit is [kg/m^3]. While not directly a "heat transfer" property like λ and Cp, it's crucial for understanding the amount of mass present to store energy. In conduction, density affects the thermal inertia of a material, i.e. the amount of matter available to heat up or cool down. A denser material will generally have more mass in a given volume, which means that it will requires more total energy to achieve a certain temperature change if its specific heat capacity is comparable.

Density also varies with temperature and pressure, especially for gases and liquids, but for solids, it's often considered relatively constant over typical operating temperature ranges.

Thermal Diffusivity

The combination of these three properties leads to a critically important derived property: thermal diffusivity (α). It's defined as the ratio between thermal conductivity and the product of the material's density (ρ) and its specific heat capacity (C_p). Thermal diffusivity measures the rate at which thermal energy diffuses or propagates through a material. It's the ratio of thermal conductivity to the heat capacity per unit volume. Materials with high thermal diffusivity respond quickly to changes in temperature at their boundaries because heat penetrates more quickly. Conversely, materials with low thermal diffusivity will take longer to undergo temperature changes throughout their volume. This property is particularly essential in transient (time-dependent) conduction problems, as it directly influences the rate of temperature propagation within a solid.

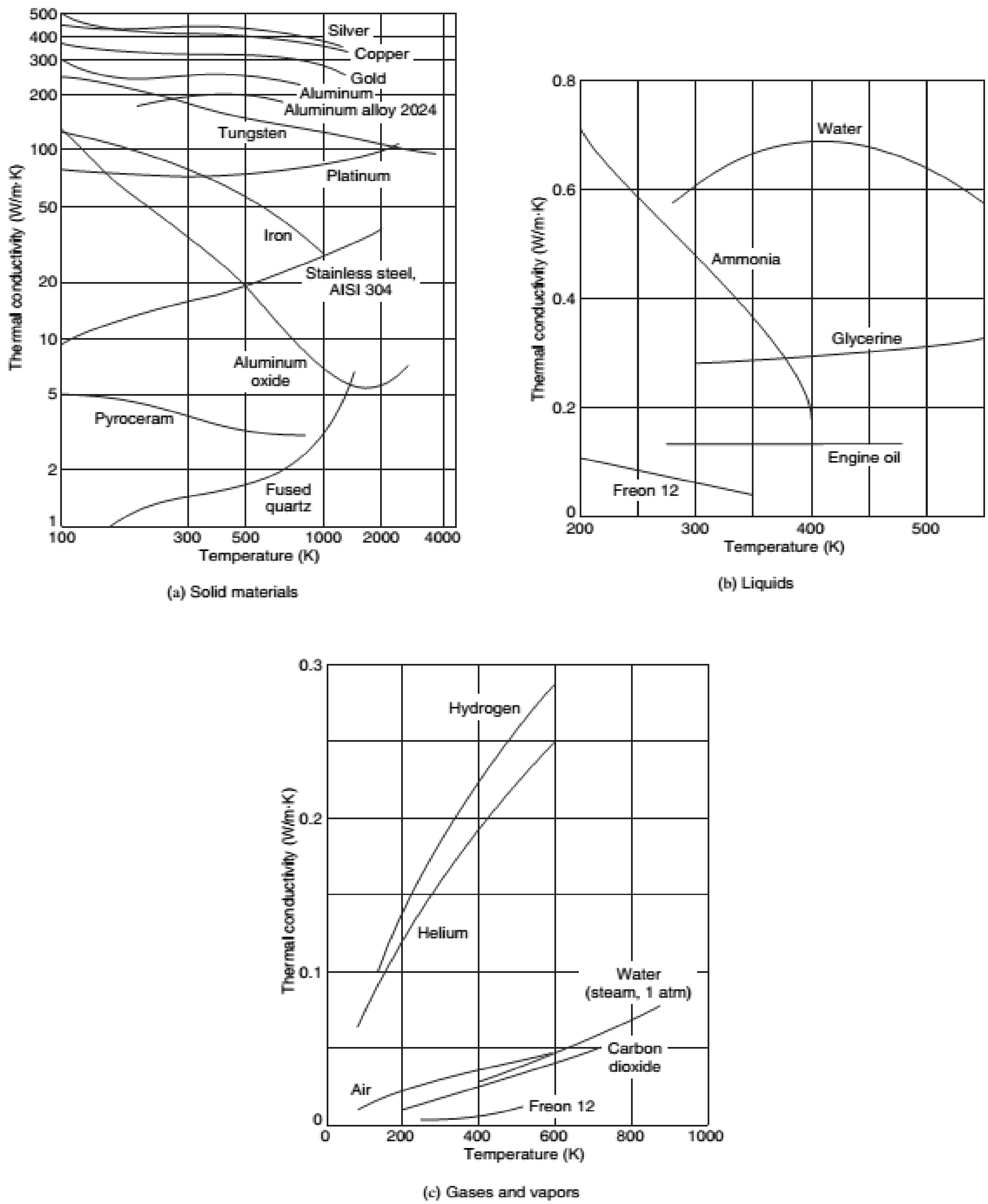


Figure 1.5: Thermal conductivity of several materials at various temperatures**.**

Table1.1: Thermal conductivity of different materials.

Materials	Thermal conductivity (W/m.°C)
Metals	
Silver (pure)	410
copper(pure)	385
Aluminum (pure)	202
Nickel(pure)	93
Iron (pure)	73
Carbon steel,1%	43
Lead (pure)	35
Chrome Nickel Steel (18% Cr,8%Ni)	16,3
Nonmetallic Solids	
Diamond	2300
Magnesite	4,15
Sandstone	1,83
Glass, window	0.78
Maple or oak	0.17
Hard rubber	0,15
Styrofoam	0.033
Liquid	
Mercury	8,21
Water	0,556
Ammonia	0,54
Freon 12, CCl2F2	0,073
Gases	
Hydrogen	0,175
Helium	0,141
Air	0,024
Water vapor (saturated)	0,0206
Carbon dioxide	0,0146

1.7 HEAT DIFFUSION EQUATION

A major objective in heat conduction analysis is to determine the temperature field within a material under specified boundary conditions. This spatial temperature distribution allows us to calculate the conductive heat flux at any point within the medium or on its surface, as defined by Fourier's law. To determine the temperature distribution, we will apply the energy conservation principle. This involves establishing a differential control volume, identifying all relevant energy transfer processes, and applying their corresponding rate equations. This process yields a differential equation which, when solved with the appropriate boundary conditions, describes the temperature distribution within the medium.

We'll now consider a homogeneous medium where convection is absent and the temperature field T(x,y,z) is described using cartesian coordinates. The initial step in applying the energy conservation method is to define an elementary control volume with dimensions of dx·dy·dz, as illustrated in figure 1.6.

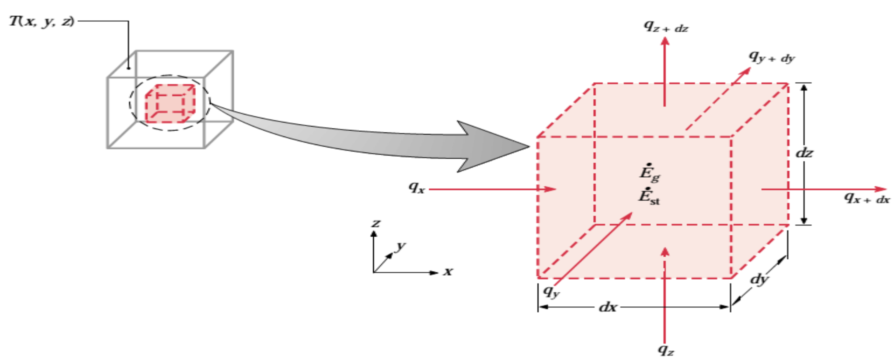


Figure 1.6: Differential control volume, dx.dy.dz, for conduction analysis in cartesian coordinates .

We apply the first law of thermodynamics at a given instant by considering the energy transfers within our defined control volume. Due to the presence of temperature gradients, heat conduction occurs across each surface. We can represent the conduction heat rates perpendicular to the surfaces at positions x, y, and z by q_x, q_y, and q_z. The description of the rates corresponding to the opposite surfaces can be carried out using a Taylor series expansion, where we simplify the expressions by neglecting higher-order terms.

$$
\begin{aligned}
q_{x+dx} &= q_x + \frac{\partial q_x}{\partial x}dx \\
q_{y+dy} &= q_x + \frac{\partial q_y}{\partial y}dy \\
q_{z+dz} &= q_z + \frac{\partial q_z}{\partial z}dz
\end{aligned}
\tag{1.2}
$$

Equation 1.2 expresses that the rate of heat transfer in the x-direction at the point (x+dx) corresponds to that at the original point (x), with the addition of an infinitesimal variation.

This adjustment is quantified by the product of the change rate of heat transfer with respect to x and the incremental distance dx. It is also necessary to take into account the internal heat generation within the medium represented by a source term .

$$\dot{E}_g = \dot{q}\,dxdydz \tag{1.3}$$

The term $\dot{q}$ represents the internal volumetric heat generation rate within the medium, expressed in (W/m^3). Furthermore, the internal thermal energy stored in the control volume can also vary. For materials that do not undergo a phase change, latent heat is not considered, and the stored energy term is given by:

$$\dot{E}_{st} = \rho C_p \frac{\partial T}{\partial t}dxdydz \tag{1.4}$$

Where $\rho C_p \frac{\partial T}{\partial t}$ represents the rate of thermal energy change per unit volume.

It's important to note that the volumetric energy generation term $\dot{q}$ in heat transfer equations represents the conversion of other energy forms into thermal energy within a medium. This is a fundamentally different process from energy transfer mechanisms like conduction, convection, and radiation, which describe the movement of thermal energy from one location to another.

The energy generation term can be positive, indicating a source of heat (e.g., from an exothermic chemical reaction, electrical resistance heating, or nuclear fission). Conversely, a negative value represents an energy sink, where thermal energy is consumed, as is the case in certain endothermic chemical reactions. Unlike energy generation, the stored energy term quantifies the rate at which a material's internal thermal energy increases or decreases.

The final step of the energy conservation method is to combine all the rate equations to express the fundamental principle of energy conservation. In terms of rates, this principle is generally expressed as:

Rate of energy stored= Rate of energy in + Rate of energy generated - Rate of energy out

Mathematically, this can be written as :

$$\dot{E}_{st} = \dot{E}_{in} - \dot{E}_{out} + \dot{E}_{g} \tag{1.5}$$

Identifying the conduction rates as the energy input ,q_i, and the output, q_s, and then substituting equations 1.2 and 1.3 in equation 1.5 , we obtain :

$$q_x + q_y + q_z - q_{x+dx} - q_{y+dy} - q_{z+dz} + \dot{q}\, dxdydz = \rho C_p \frac{\partial T}{\partial t} dxdydz \tag{1.6}$$

Substituting from equations 1.2, it follows that:

$$-\frac{\partial q_x}{\partial x}dx-\frac{\partial q_y}{\partial y}dy-\frac{\partial q_z}{\partial z}dz+\dot{q}\,dxdydz=\rho C_p\frac{\partial T}{\partial t}dxdydz \tag{1.7}$$

The conduction heat rates may be evaluated from Fourier's law:

$$\begin{aligned} q_x &= -\lambda\frac{\partial T}{\partial x} \\ q_y &= -\lambda\frac{\partial T}{\partial y} \\ q_z &= -\lambda\frac{\partial T}{\partial z} \end{aligned} \tag{1.8}$$

To obtain the heat transfer rates, each heat flux term in Equation 1.8 is multiplied by the appropriate surface area of the infinitesimal control volume. Subsequently, by substituting these expressions into the energy balance (equation 1.7), and dividing the entire equation by the control volume (dx·dy·dz), we find:

$$\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(\lambda\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(\lambda\frac{\partial T}{\partial z}\right)+\dot{q}=\rho C_p\frac{\partial T}{\partial t} \tag{1.9}$$

Equation 1.9 is the general heat diffusion equation in cartesian coordinates, it represents the main equation for analyzing heat conduction. By solving it, we can determine the temperature at any point in material (x,y,z) and at any time. Despite its apparent complexity, the equation's fundamental meaning is a simple statement of the principle of energy conservation. When a material's thermal conductivity (λ) is assumed to be constant, the general heat diffusion equation (equation 1.9) simplifies to:

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{q}}{\lambda}=\frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{1.10}$$

Where $\alpha = \dfrac{\lambda}{\rho C_p}$ is the thermal diffusivity.

The general form of the heat equation can often be further simplified. For example, in a steady-state system, the rate of energy stored is zero. Consequently, equation 1.9 simplifies to the following :

$$\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(\lambda\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(\lambda\frac{\partial T}{\partial z}\right)+\dot{q}=0 \tag{1.11}$$

Moreover, under the specific conditions of one-dimensional heat transfer (for instance, along the x-axis) and the absence of energy generation, equation 1.11 simplifies significantly to:

$$\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right)=0 \tag{1.12}$$

An important consequence of this result is that when heat transfer is steady, occurs in only one direction, and there's no internal energy generation, the rate of heat transfer per unit area (heat flux) is uniform throughout the heat path. The heat equation is not limited to cartesian coordinates and can also be expressed using cylindrical and spherical coordinate systems .

1.8 CONDUCTION EQUATION IN RADIAL GEOMETRY

Equation (1.10) may be transformed into either cylindrical or spherical coordinates by standard calculus techniques. The results are as follows:

<u>In cylindrical coordinates</u>

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{q}}{\lambda}=\frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{1.13}$$

In spherical coordinates

$$\frac{\partial^2 T}{\partial r^2}+\frac{2}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2}+\frac{\dot{q}}{\lambda}=\frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{1.14}$$

The general heat conduction equations in cartesian, cylindrical, and spherical coordinates can be simplified significantly under common physical assumptions relevant to practical applications. These simplifications reduce mathematical complexity and allow us to focus on the dominant modes of heat transfer.

1.9 BOUNDARY AND INITIAL CONDITIONS.

Boundary (surface) conditions:

The most frequently encountered boundary conditions in conduction are as follows:

A. Dirichlet Boundary Condition (Constant surface Temperature):

The surface temperature of the boundaries is specified as either a constant value. This type of boundary condition is known as a Dirichlet boundary condition. Generally, it is expressed by

$$\begin{aligned} T(0,t) &= T_0 \\ T(L,t) &= T_L \end{aligned} \tag{1.15}$$

B. Neumann Boundary Condition (Constant heat flux):

The heat flux boundary condition specifies the rate of heat transfer across the system's edges, which may be constant or vary with location and/or time, this condition dictates that the net heat flux at a boundary must be zero. We adopt the convention that heat flux entering a boundary is positive, while heat flux leaving is negative.

Thus, remembering that the statement of Fourier's law is independent of the choice of direction for the heat flux vector q_n, we can conveniently define q_n such that it is positive in the outward normal direction from the surface. Accordingly, we have from Figure 1.7:

$$\pm\lambda\frac{\partial T}{\partial n} = \pm q''_n \tag{1.16}$$

where $\frac{\partial}{\partial n}$ denotes differentiation along the normal of the boundary. The plus and minus signs of the left-hand side of equation (1.16) correspond to the differentiations along the inward and outward normal, respectively, and the plus or minus signs on the right-hand side correspond to the direction of heat flux: positive when heat leaves the boundary (outward), and negative when heat enters the boundary (inward) .

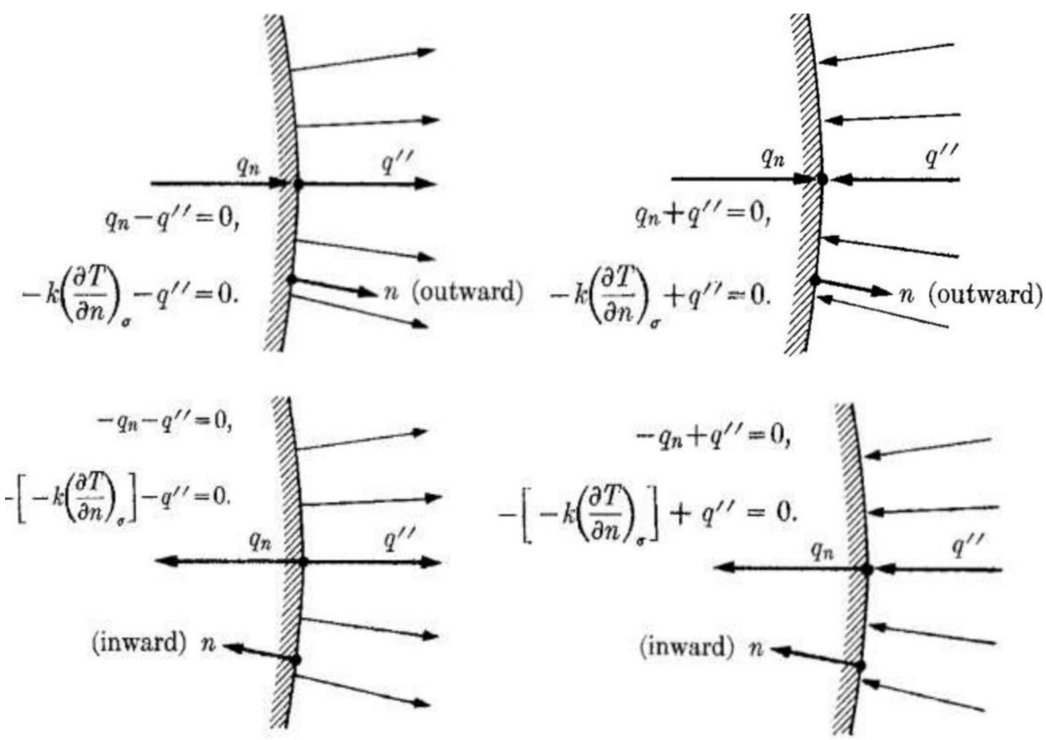


Figure1.7: Constant heat flux boundary conditions.

C. No heat flux (insulation)

A special case of the heat flux boundary condition is the insulated or adiabatic surface, which is obtained by setting the heat flux to zero $q^{"}_{n} = 0$ in equation (1.16). This yield:

$$\frac{\partial T}{\partial n} = 0 \tag{1.17}$$

D. Robin Boundary Condition (Convection):

When the heat flux across the boundaries of a continuous body is not known, it's often assumed to be proportional to the temperature difference between the body's surface and the surrounding environment. This is described by the Robin boundary condition, which is expressed by the following formula:

$$\pm\lambda \frac{\partial T}{\partial n} = h\left(T - T_{\infty}\right) \tag{1.18}$$

Where T is the temperature of the solid boundaries, T_{∞} is the temperature of the surrounding environment at a distance far from the boundaries, and the constant h, is known as the heat transfer coefficient.

The plus and minus signs of the left member of equation (1.18) correspond the direction of the temperature gradient (Figure 1.8).

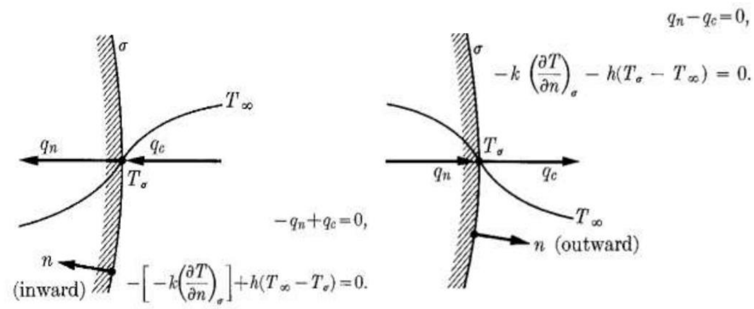


Figure.1.8: Heat transfer by convection surface heat flux boundary conditions .

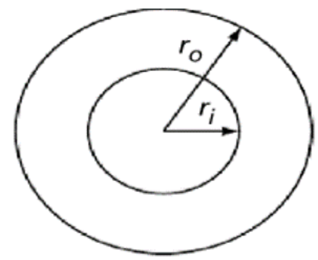
E. Heat transfer to the ambient by radiation and convection

To model heat transfer via radiation from the boundaries of a medium, a specific boundary condition is necessary. Assuming a uniform but unknown temperature T_1 for the medium, the boundary condition can be formulated to account for the combined effects of conductive and radiative heat transfer at the surface. The net heat flux is given by the sum of these two components, as described by the following expression:

$$\pm\lambda \frac{\partial T}{\partial n} = \varepsilon\sigma\left(T_1^4 - T_{surr}^4\right) + h\left(T_1 - T_{surr}\right) \quad (1.19)$$

EXERCISES

EXERCISE 1.1



The temperature distribution in along cylindrical tube is:

$$T\left(r\right)=800+1200r-3000r^{2}$$

Where T is in Kelvin and r is in meter. The cylindrical tube has inner radius of 25 cm and outer radius of 40 cm,its thermal conductivity is 50 [W/(m.K)

-Find the rate of heat transfer entering and leaving the cylinder.

Solution

$$T\left(r\right)=800+1200r-3000r^{2}$$
$$\frac{\partial T}{\partial r}=1200-6000r$$

Hence, the rate of heat transfer entering per unit length is:

$$\phi_r=-\lambda\frac{\partial T}{\partial r}A=-50.\left(1200-6000r\right)\times 1$$

$$\phi_{r=0.25}=-\lambda\left.\frac{\partial T}{\partial r}\right|_{0,25}A=$$
$$=-50\left(2\pi.0,25\right)\left(1200-6000\left(0,25\right)\right)\times 1$$
$$=23500\left[W\right]$$

Similarly, the rate of heat transfer leaving the system per unit length is:

$$\phi_{r=0,4}=-\lambda\left.\frac{\partial T}{\partial r}\right|_{0,4}A$$

$$\phi_{r=0,4} = -50\left(2\pi \times 0,4\right)\left(1200 - 6000\left(0,4\right)\right)$$
$$= 150720\left[W\right]$$

EXERCISE 1.2

The temperature distribution in a plane wall of 50 cm thick at a given time is expressed by the relation: $T\left(x\right) = 450 - 500x + 100x^2 + 150x^3$

Where T is the temperature in °C and x in meters. The thermal conductivity of the wall material is 10 [W/m.K].

-Calculate the rate of heat energy stored per unit area of the wall at this instant.

Solution

$$T\left(x\right) = 450 - 500x + 100x^2 + 150x^3$$
$$\frac{dT}{dx} = -500 + 200x + 450x^2$$

Heat entering the wall from the face being heated i.e x=0 is

$$\phi_{in} = -\lambda \left.\frac{dT}{dx}\right|_{x=0} A$$
$$\phi_{in} = -10 \times 1 \times \left.\left(-500 + 200x + 450x^2\right)\right|_{x=0}$$
$$= -10\left(-500\right) = 5000\left[W\right]$$

Heat leaving the wall i.e at x=0,5 is:

$$\phi_{out} = -\lambda \left.\frac{dT}{dx}\right|_{x=0,5} A$$

$$\phi_{out} = -10 \times 1 \times \left(-500 + 200x + 450x^2\right)\Big|_{x=0,5}$$
$$= -10\left(-500 + 100 + 112.5\right) = 2875[W]$$

$$\phi_{stored} = \phi_{in} - \phi_{out}$$

$$\phi_{stored} = 5000 - 2875 = 2125W$$

EXERCISE 1.3

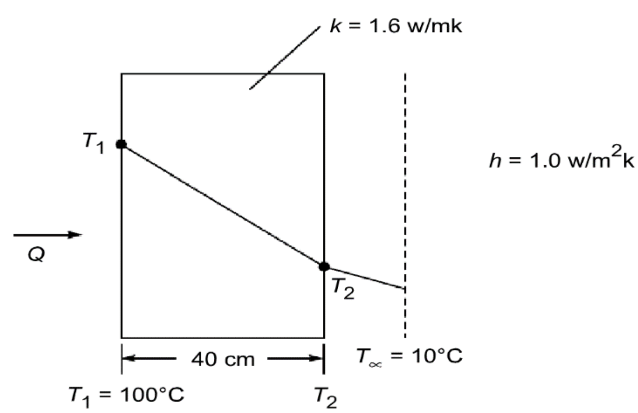
One side of a plane wall is maintained at 100°C while other side is exposed to a convection environment having T_∞=10 [°C] and h = 10 [W/ (m^2 .K)]. This wall has the dimensions of 3x5 m^2, a thermal conductivity of 1,6 [W/(m. K)] and the thickness of 40cm.

-Calculate the heat transfer rate through the wall.

Solution

We have: $A = 3\ (5) = 15\ m^2$

The heat transfer rate is:



$$\phi = -\lambda \frac{dT}{dx} A = hA\left(T_2 - T_\infty\right)$$
$$Or \quad 1,6\frac{T_1 - T_2}{0.4} = 10\left(T_2 - 10\right)$$

$$T_1 - T_2 = \frac{10}{4}\left(T_2 - 10\right)$$
$$Or \quad 3.5T_2 = \left(T_1 + 25\right)$$

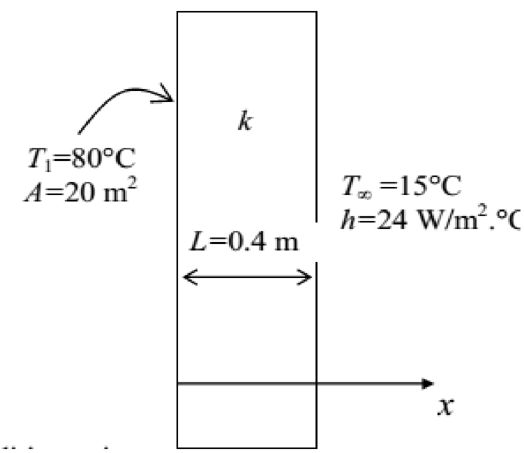
$$T_2 = \frac{125}{3.5} = 35.714°C$$

Thus $\phi = 1,6\left(3\times 5\right)\dfrac{\left(100-35,714\right)}{0,4} = 3857.16\left[W\right]$

EXERCISE 1.4

A large plane wall is subjected to specified temperature $\left(T(0)=T_1=80°C\right)$ on the left surface and convection characterised by $T_\infty = 15°C$ and $h=24\left[W/(m^2.°C)\right]$ on the right surface.

Find the mathematical formulation of the temperature variation, and the rate of heat transfer for steady one-dimensional heat transfer.



Data: $L=0,4m, A=20\left[m^2\right]$ and $\lambda = 50\left[W/(m.K)\right]$

<u>Solution</u>

$$\frac{d^2T}{dx^2}=0$$

and $T(0)=T_1=80°C$

$$-\lambda\frac{dT}{dx}=h\left(T(L)-T_\infty\right)$$

1°/ Integrating the differential equation twice with respect to x yields

$$\frac{dT}{dx}=C_1$$

$$T(\mathrm{x})=C_1x+C_2$$

where C_1 and C_2 are arbitrary constants. Applying the boundary conditions give

$$\left.\begin{array}{l}\text{At } x=0: T(0)=C_2=T_1\\ \text{At } x=L: -\lambda C_1 = h\left[\left(C_1L+C_2\right)-T_\infty\right]\end{array}\right\}\Rightarrow\left\{\begin{array}{l}C_2=T_1\\ C_1=\dfrac{h\left(T_\infty-C_2\right)}{\lambda+hL}=\dfrac{h\left(T_\infty-T_1\right)}{\lambda+hL}\end{array}\right.$$

2°/ Substituting C_1 and C_2 into the general solution, we get the temperature variation as:

$$T\left(x\right)=\frac{h\left(T_{\infty}-T_{1}\right)}{\lambda+hL}x+T_{1}=\frac{24\left(15-80\right)}{50+\left(24\times 0,4\right)}x+80$$
$$=-26{,}174x+80$$

3°/ The rate of heat conduction through the wall is:

$$\phi=-\lambda\frac{dT}{dx}A=131{,}1\lambda A$$
$$=(26{,}174)\ (50)\ 20$$
$$=26174\left[W\right]$$

CHAPTER 2: ONE-DIMENSIONAL STATIONARY CONDUCTION

2.1 INTRODUCTION

The study of one-dimensional (1D) heat conduction is fundamental in thermal engineering, as it models heat transfer in highly symmetric objects: the plane walls, cylinders, and the spheres where the heat flow is dominant along a single spatial axis. This process is governed by the heat diffusion equation.

We consider two main cases: the model without an internal heat source, which describes simple transfer through the medium, and the model including a source term, which is necessary to analyze systems with internal energy generation.

2.2 CONDUCTION WITHOUT INTERNAL ENERGY GENERATION

Heat conduction in one dimension simplifies the general heat equation to forms that are solvable analytically under specific boundary conditions. The choice of coordinate system cartesian, cylindrical, or spherical depends on the object geometry, such as a slab, a pipe, or a sphere. The geometry of the system significantly influences the form of the conduction equation due to the divergence operator taking different forms in each coordinate system. When there is no internal heat generation, the resulting equation is homogeneous. In contrast, the presence of internal heat sources introduces a nonhomogeneous term, complicating the solution but offering a more realistic representation of many physical scenarios. These cases are fundamental for understanding thermal diffusion in a variety of engineering and physical systems. The resulting solutions whether transient or steady-state enable the prediction of temperature distributions, which are critical for thermal design.

2.3 STEADY-STATE ONE-DIMENSIONAL HEAT CONDUCTION WITHOUT HEAT GENERATION:

2.3.1. Plan wall

Let's assume figure 2.1 shows a plane wall of thickness L, extending infinitely in the y and z directions. To ensure that the mathematical treatment is consistent with the physical behavior, the following conditions are applied:

(1) Conduction only in x-direction ⇒ $T = T(x)$ so $\frac{\partial T}{\partial y} = \frac{\partial T}{\partial z} = 0$

(2) No heat source ⇒ $\dot{q}_g = 0$ (3) Steady state ⇒ $\frac{\partial T}{\partial t} = 0$

(4) Thermal conductivity is constant $\lambda = C^{te}$

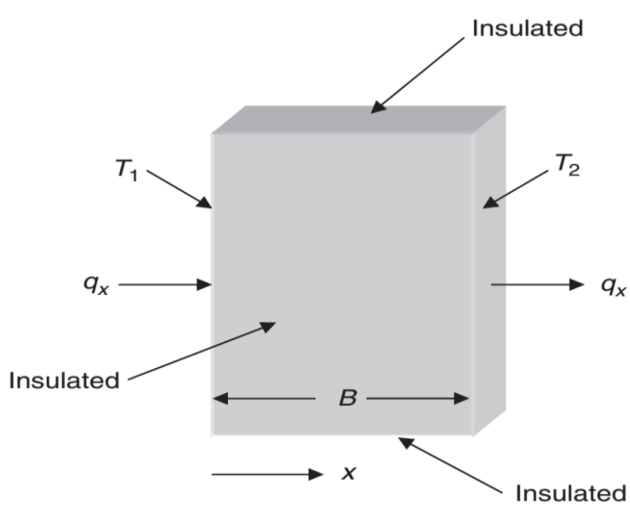


Figure 2.1: One-dimensional heat conduction in a solid.

The conduction equation in cartesian coordinates then becomes:

$$\lambda \frac{\partial^2 T}{\partial x^2} = 0 \text{ or } \frac{d^2 T}{dx^2} = 0 \tag{2.1}$$

The partial derivative is simplified to a total derivative because the equation depends only on the single independent variable, x. Subsequently, integrating both sides of this equation results in:

$$\frac{dT}{dx} = C_1 \tag{2.2}$$

A second integration gives: $T = C_1 x + C_2$ (2.3)

Thus, it is seen that the temperature varies linearly across the solid. The constants of integration can be determined by applying the boundary conditions:

$$\begin{aligned} &At\ x = 0 \quad \Rightarrow T = T_1 \\ &At\ x = L \quad \Rightarrow T = T_2 \end{aligned} \tag{2.4}$$

The first boundary condition gives $C_2=T_1$ and the second then gives :

$$T_2 = C_1 L + T_1 \tag{2.5}$$

Solving for C_1 , we find: $C_1 = \frac{T_2 - T_1}{L}$ (2.6)

The heat flux is obtained from Fourier's law:

Therefore, $\varphi = -\lambda \frac{T_2 - T_1}{L} = \lambda \frac{T_1 - T_2}{L}$ (2.7)

Multiplying by the area gives the rate of heat conduction:

$$\phi = \lambda \frac{T_1 - T_2}{L} A \tag{2.8}$$

Composite wall in series

Considering a multilayer wall (figure 2.2), the presence of several materials necessitates an analysis that takes into account the distinct temperature gradients shown in the three materials.

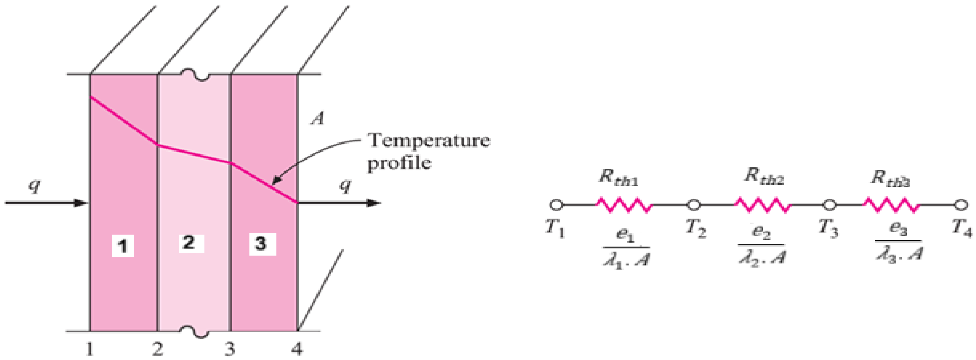


Figure 2.2: composite wall in series

The steady-state heat flux can then be expressed by:

$$\phi = -\lambda_1 \frac{T_2 - T_1}{e_1} A = -\lambda_2 \frac{T_3 - T_2}{e_2} A = -\lambda_3 \frac{T_4 - T_3}{e_3} A \tag{2.9}$$

The thermal resistance by definition is: $\mathrm{R}_{th} = \frac{\Delta T}{\phi}$.

Therefore $$\phi = \frac{T_1 - T_2}{R_{th1}} = \frac{T_2 - T_3}{R_{th2}} = \frac{T_3 - T_4}{R_{th3}} = \frac{T_1 - T_4}{R_{th-eq}} \tag{2.10}$$

Where $$R_{th1} = \frac{e_1}{\lambda_1 . A}, R_{th2} = \frac{e_2}{\lambda_2 . A}, R_{th3} = \frac{e_3}{\lambda_3 . A} \tag{2.11}$$

And $$R_{th-eq} = R_{th1} + R_{th2} + R_{th3} \tag{2.12}$$

Composite wall in parallel

Consider the composite wall shown in figure 2.3, which is composed of two parallel layers. The associated thermal resistance network can be represented by two parallel resistances, as illustrated in the flowing figure

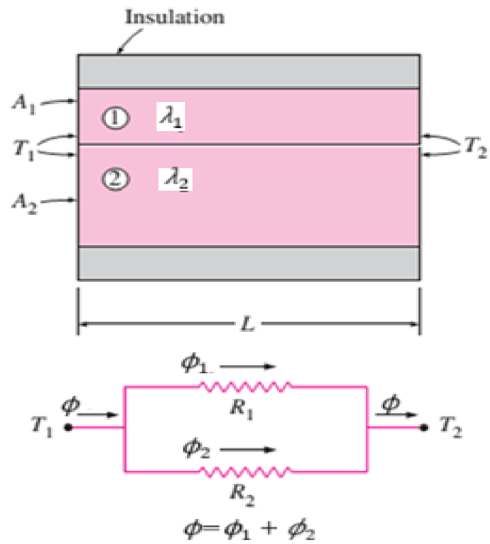


Figure 2.3: composite wall in parallel

Since the total heat transfer ϕ is the sum of the heat transfers through each individual layer (ϕ_1 and ϕ_2), the total heat transfer relationship is given by :

$$\phi = \phi_1 + \phi_2 = \frac{T_1 - T_2}{R_{th1}} + \frac{T_1 - T_2}{R_{th2}} = (T_1 - T_2)\left(\frac{1}{R_{th1}} + \frac{1}{R_{th2}}\right) = \frac{(T_1 - T_2)}{R_{th-eq}} \quad (2.13)$$

$$\frac{1}{R_{th-eq}} = \left(\frac{1}{R_{th1}} + \frac{1}{R_{th2}}\right) so\ R_{th-eq} = \left(\frac{1}{R_{th1}} + \frac{1}{R_{th2}}\right)^{-1} \quad (2.14)$$

2.3.2. Radial geometry

Radial heat conduction through a cylinder

Consider a long cylindrical geometry (figure 2.4) with an inner radius R_1 and an outer radius R_2, having a length L. Due to its significant length, we can neglect heat losses from the ends compared to the radial heat transfer. The inner and outer cylindrical surfaces are maintained at uniform temperatures T_1 and T_2.

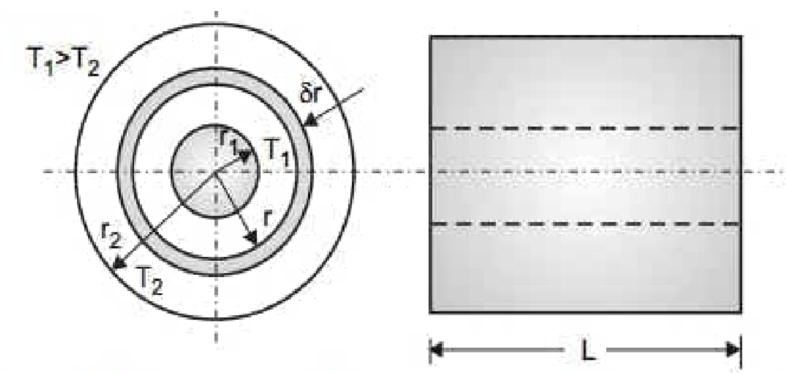


Figure 2.4: Steady state heat conduction through a cylinder.

The general heat conduction equation in cylindrical coordinates is given by:

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{\lambda} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{2.15}$$

Assumptions

-Heat conduction is only in radial direction.

-There is no heat generation within the cylinder.

-Steady state conditions i.e. temperature variation with respect to time is zero.

-Above equation takes the form .

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}=0 \tag{2.16}$$

Or $$\frac{d}{dr}\left(r\frac{dT}{dr}\right)=0 \tag{2.17}$$

The boundary conditions applied are:

At radius $r=R_1$, $T=T_1$

At radius $r=R_2$, $T=T_2$

Integrating equation 2.17 twice, we get :

$$T(r)=C_1\ln r+C_2 \tag{2.18}$$

Using the boundary conditions

$$\begin{aligned} &At\ r=R_1,\ T=T_1,\quad T_1=C_1\ln R_1+C_2 \\ &At\ r=R_2,\ T=T_2,\quad T_2=C_1\ln R_2+C_2 \end{aligned} \tag{2.19}$$

Hence

$$C_1=\frac{T_1-T_2}{\ln\left(\frac{R_1}{R_2}\right)} \tag{2.20}$$

$$C_2=T_1-\frac{T_2-T_1}{\ln\left(\frac{R_2}{R_1}\right)}\ln R_1 \tag{2.21}$$

Substituting the value of C_1 and C_2 in equation 2.18, we obtain:

$$\mathrm{T(r)} = \frac{T_2 - T_1}{\ln\left(\frac{R_2}{R_1}\right)} \ln\frac{r}{R_1} + T_1 \tag{2.22}$$

So, the rate of heat transfer is:

$$\phi = -\lambda \frac{dT}{dr} A_r = -\lambda \frac{T_2 - T_1}{\ln\left(\frac{R_2}{R_1}\right)} \left(\frac{1}{r}\right) A_r \tag{2.23}$$

where $A_r = 2\pi r L$

Therefore $\phi = \dfrac{T_1 - T_2}{\ln\left(\frac{R_2}{R_1}\right)} 2\pi\lambda L$ (2.24)

The thermal resistance in this case is: $\mathrm{R_{th}} = \dfrac{1}{2\pi\lambda L} \ln\left(\dfrac{R_2}{R_1}\right)$ (2.25)

Multi-layered cylinders and spheres

For three cylindrical resistances in series (figure 2.5)

$$\mathrm{R}_{th-eq} = \mathrm{R}_{th1} + \mathrm{R}_{th2} + \mathrm{R}_{th3} \tag{2.26}$$

Where $\mathrm{R}_{th1} = \dfrac{\ln\left(R_2 / R_1\right)}{2\lambda_1 L\pi}, \mathrm{R}_{th2} = \dfrac{\ln\left(R_3 / R_2\right)}{2\lambda_2 L\pi}, \mathrm{R}_{th3} = \dfrac{\ln\left(R_4 / R_3\right)}{2\lambda_3 L\pi}$ (2.27)

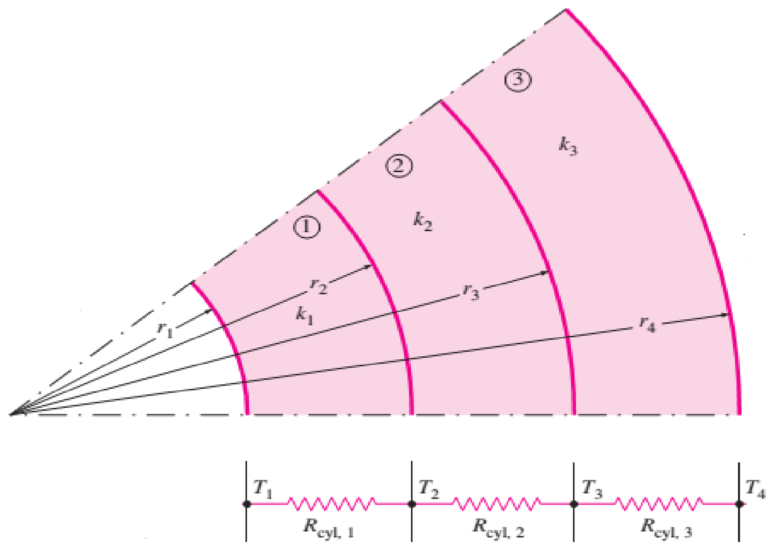


Figure 2.5: The thermal resistance network for heat transfer through a composite cylinder.

The rate of heat transfer is:

$$\phi = \left(\frac{T_1 - T_2}{R_{th1}}\right) = \left(\frac{T_2 - T_3}{R_{th2}}\right) = \left(\frac{T_3 - T_4}{R_{th3}}\right) = \left(\frac{T_1 - T_4}{R_{th1} + R_{th2} + R_{th3}}\right) = \left(\frac{T_1 - T_4}{R_{th-eq}}\right) \tag{2.28}$$

Combining the convection (internal and external) conditions, We obtain:

$$\mathrm{R}_{th-in} = \frac{1}{h_1 \pi D_1 L}, \mathrm{R}_{th-out} = \frac{1}{h_2 \pi D_2 L} \tag{2.29}$$

Then $\phi = \left(\dfrac{T_{\infty 1} - T_{\infty 2}}{R_{th-in} + R_{th-1} + R_{th-2} + R_{th-3} + R_{th-out}}\right)$ (2.30)

Radial heat conduction through a sphere .

Consider a quarter spherical section represented in figure 2.6.

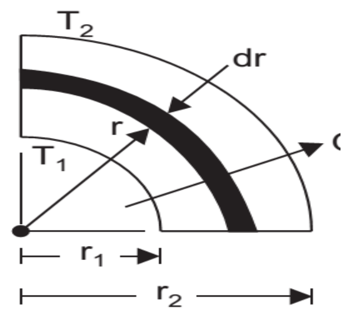


Figure 2.6. Radial heat conduction-hollow sphere.

The heat equation in this case is expressed by:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)=0 \qquad (2.31)$$

Hence $r^2\frac{\partial T}{\partial r}=C_1$ Or $\frac{dT}{dr}=\frac{C_1}{r^2}$ (2.32)

Integrating this equation, we obtain: $T=-\frac{C_1}{r}+C_2$ (2.33)

For the boundary condition: $T\left(r=R_1\right)=T_1$ and $T\left(r=R_2\right)=T_2$,we get:

$$\begin{cases} T_1=-\frac{C_1}{R_1}+C_2 \\ T_2=-\frac{C_1}{R_2}+C_2 \end{cases} \qquad (2.34)$$

Therefore :

$$T_1-T_2=C_1\left(\frac{1}{R_2}-\frac{1}{R_1}\right) \text{ Or } C_1=\frac{T_1-T_2}{\frac{1}{R_2}-\frac{1}{R_1}} \qquad (2.35)$$

$$C_2 = T_1 + \frac{C_1}{R_1} = T_1 + \frac{T_1 - T_2}{\frac{1}{R_2} - \frac{1}{R_1}} \frac{1}{R_1} \tag{2.36}$$

And ,

Substituting in equation (2.33), we obtain:

$$T = -\frac{T_1 - T_2}{\frac{1}{R_2} - \frac{1}{R_1}} \frac{1}{r} + T1 + \frac{T_1 - T_2}{\frac{1}{R_2} - \frac{1}{R_1}} \frac{1}{R1} \tag{2.37}$$

The thermal resistance in this case is: $\frac{1}{4\pi\lambda}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$. (2.38)

Conduction through composite sphere

For three spherical resistances in series (figure 2.7)

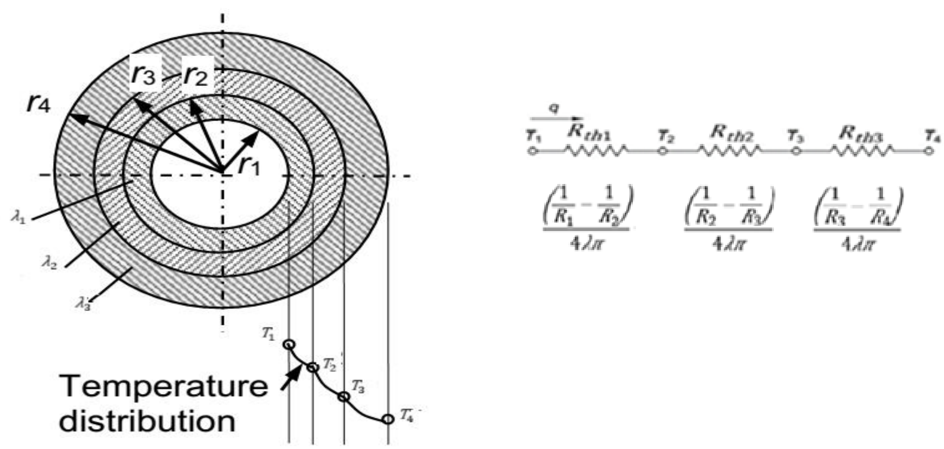


Figure 2.7: Conduction through composite sphere

$$R_{th-eq} = R_{th1} + R_{th2} + R_{th3} \tag{2.39}$$

Where $R_{th1} = \frac{\left(\frac{1}{R_1} - \frac{1}{R_2}\right)}{4\lambda\pi}, R_{th2} = \frac{\left(\frac{1}{R_2} - \frac{1}{R_3}\right)}{4\lambda\pi}, R_{th3} = \frac{\left(\frac{1}{R_3} - \frac{1}{R_4}\right)}{4\lambda\pi}$ (2.40)

The rate of heat conduction in this case is given by :

$$\phi = \left(\frac{T_1 - T_2}{R_{th1}} \right) = \left(\frac{T_2 - T_3}{R_{th2}} \right) = \left(\frac{T_3 - T_4}{R_{th3}} \right) = \left(\frac{T_1 - T_4}{R_{th-eq}} \right) \tag{2.41}$$

$$\phi = \left(\frac{T_1 - T_4}{\frac{1}{4\pi\lambda_1}\left(\frac{1}{R_1} - \frac{1}{R_2} \right) + \frac{1}{4\pi\lambda_2}\left(\frac{1}{R_2} - \frac{1}{R_3} \right) . + \frac{1}{4\pi\lambda_3}\left(\frac{1}{R_3} - \frac{1}{R_4} \right)} \right) = \frac{T_1 - T_4}{R_{th-eq}} \tag{2.42}$$

Combining the convection (internal and external) conditions, we obtain:

$$\mathrm{R}_{th-in} = \frac{1}{h_1 \pi {D^2}_1}, \mathrm{R}_{th-out} = \frac{1}{h_2 \pi {D^2}_2} \tag{2.43}$$

Then $\phi = \left(\frac{T_{\infty 1} - T_{\infty 2}}{R_{th-in} + R_{th-1} + R_{th-2} + R_{th-3} + R_{th-out}} \right)$ (2.44)

2.4 CONDUCTION WITH INTERNAL ENERGY GENERATION

Many applications in the field of heat transfer involve internal heat generation, such as in electrical conductors, nuclear reactors, and chemical processes. In this study, we focus on one-dimensional systems, where the temperature distribution varies as a function of a single spatial coordinate.

A. <u>Steady state radial heat conduction in plan wall</u>

Consider a plane wall of thickness 2L, as illustrated in figure 2.8, with large dimensions in the other directions. This geometric configuration justifies the assumption of one-dimensional heat transfer along the x-axis. The wall is subject to a uniform internal heat generation rate per unit volume $\dot{q}$, and its thermal conductivity λ is assumed to be constant throughout the material.

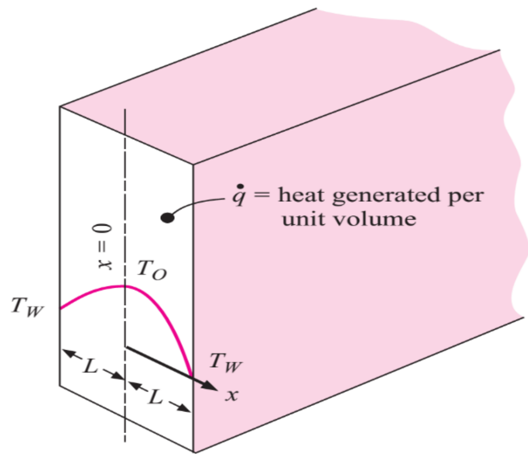


Figure 2.8: One-dimensional conduction problem with heat generation.

The differential equation that governs the temperature distribution is:

$$\frac{d^2T}{d^2x}+\frac{\dot{q}}{\lambda}=0 \tag{2.45}$$

For the boundary conditions, we specify the temperatures on either side of the wall, i.e.,

$$T = T_w \qquad \text{at } x=\pm L \tag{2.46}$$

The general solution of equation (2.45) is

$$T(x)=-\frac{\dot{q}}{2\lambda}x^2+C_1x+C_2 \tag{2.47}$$

Since the temperature is equal on both surfaces of the wall, the constant of integration C_1 must be zero $C_1 = 0$. The temperature at the wall's midplane (x=0) is defined by T_0. According to equation (2.47), we can obtain $C_2 = T_0$

The temperature distribution is therefore.

$$T(x) - T_0 = -\frac{\dot{q}}{2\lambda} x^2 \tag{2.48.a}$$

For x=L, we have: $T(x = L) = T_w = T_0 - \frac{\dot{q}}{2\lambda} L^2$ (2.48.b)

Therefore:

$$\frac{T - T_0}{T_w - T_0} = \left(\frac{x}{L}\right)^2 \tag{2.48.c}$$

This parabolic temperature distribution allows us to determine the midplane temperature, T_0, by applying the principle of energy conservation. At steady state, the total rate of heat generation within the wall must be equal to the total rate of heat loss from its surfaces.

Consequently: $-2\lambda \left.\frac{dT}{dx}\right|_L A = 2AL\dot{q}$ (2.49)

Where A is the cross-sectional area of the plate. The temperature gradient at the wall is obtained by differentiating equation (2.48.c):

$$\left.\frac{dT}{dx}\right|_L = (T_w - T_0)\left.\frac{2x}{L^2}\right|_{x=L} = (T_w - T_0)\frac{2}{L} \tag{2.50}$$

Then $-\lambda(T_w - T_0)\frac{2}{L} = \dot{q}L$ (2.51)

So $T_0 = T_w + \frac{\dot{q}}{2\lambda} L^2$ (2.52)

The equation for the temperature distribution could also be written in the alternative form

$$\frac{T-T_w}{T_0-T_w}=1-\left(\frac{x}{L}\right)^2 \qquad (2.53)$$

B. Steady state radial heat conduction in cylinder with heat generation

Let's consider a cylinder of radius R undergoing to uniform heat generation $\dot{q}$ and having a thermal conductivity λ. Its outer surface is exposed to convection characterized by h and T_∞

To analyze the heat transfer, we perform an energy balance on a small annular volume within the cylinder, located between radius r and r+dr, considering a unit length of 1 m. This balance is expressed as:

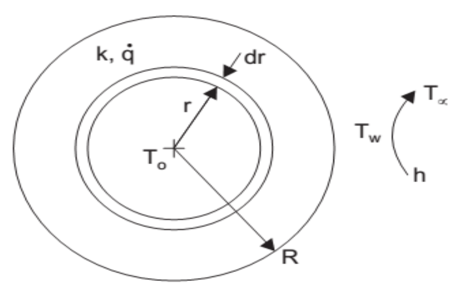
heat conducted inlet + heat generated – heat conducted outlet =0

Therefore $$\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{\dot{q}}{\lambda}r=0 \qquad (2.54)$$

Integrating after separating variables

$$\frac{dT}{dr}=-\frac{\dot{q}}{2\lambda}r+\frac{C_1}{r}$$
$$T(r)=-\frac{\dot{q}}{4\lambda}r^2+C_1\ln r+C_2 \qquad (2.55)$$

This represents the general solution to the cylindrical problem with the heat source. The values of the constants C_1 and C_2 are determined by applying the appropriate boundary conditions.



For solid cylinder: The boundary conditions are:

$$\text{At } r = 0, \frac{dT}{dr} = 0$$
$$\text{At } r = R, T = T_w \tag{2.56}$$

The first condition yields: $C_1 = 0$ (2.57.a)

The second condition yields: $C_2 = T_w + \frac{\dot{q}}{4\lambda} R^2$ (2.57.b)

Therefore $T - T_w = \frac{\dot{q}}{4\lambda}\left(R^2 - r^2\right)$ (2.58)

The maximum temperature, T_{max} is obtained for r=0, it is equal to:

$$T_{\max} - T_w = \frac{\dot{q}}{4\lambda} R^2 \tag{2.59}$$

Therefore $\frac{T - T_w}{T_{\max} - T_w} = 1 - \left(r / R\right)^2$ (2.60)

The temperature distribution within the cylinder varies parabolically with the radius .

Taking convection into account, heat generation unit length is $\dot{q}.\pi R^2.1$

This is absorbed by the fluid in the outside area $2\pi R.1$

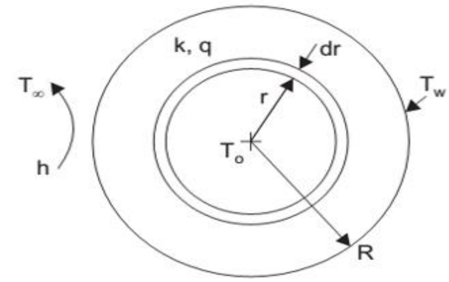
Equation (2.58) reduces to $T - T_\infty = \frac{\dot{q}}{4\lambda}\left(R^2 - r^2\right) + \frac{\dot{q}R}{2h}$ (2.61)

Equation (2.59) reduces to $T_{\max} = T_\infty + \frac{\dot{q}}{4\lambda} R^2 + \frac{\dot{q}R}{2h}$ (2.62)

C. Radial conduction in sphere with heat generation

Applying the energy equation to a thin layer of thickness dr positioned at radius r, we obtain:

$$-4\lambda\pi r^2 \frac{dT}{dr} + 4\pi r^2 \dot{q}\, dr + 4\lambda\pi r^2 \frac{dT}{dr} + \frac{d}{dr}\left(4\lambda\pi r^2 \frac{dT}{dr}\right) dr = 0$$



$$\frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) + \frac{\dot{q}\, r^2}{\lambda} = 0 \qquad (2.63)$$

Integrating this equation gives :

$$\frac{dT}{dr} = -\frac{\dot{q}}{3\lambda} r + \frac{C_1}{r^2} \qquad (2.64)$$

At $r = 0,\quad \dfrac{dT}{dr} = 0$

Hence: $$C_1 = \frac{\dot{q}}{3\lambda} r^3 = 0 \qquad (2.65)$$

Therefore

$$T = -\frac{\dot{q}}{6\lambda} r^2 + C_2 \qquad (2.66)$$

At $r = R : T = T_w$

Thus $$C_2 = T_w + \frac{\dot{q}}{6\lambda} R^2 \qquad (2.67)$$

Hence: $$T - T_w = \frac{\dot{q}}{6\lambda}\left(R^2 - r^2\right) \qquad (2.68)$$

The maximum temperature is calculated for r=0

$$T_0 - T_w = T_{max} - T_w = \frac{\dot{q}}{6\lambda} R^2 \quad (2.69)$$

Therefore

$$\frac{T - T_w}{T_{max} - T_w} = 1 - (r/R)^2 \quad (2.70)$$

Therefore $1 - \frac{T - T_w}{T_0 - T_w} = \frac{T_0 - T}{T_0 - T_w} = (r/R)^2$ (2.71)

Considering convection, the energy balance at the outside is .

$$\frac{4}{3}\pi R^3 \dot{q} = 4\pi R^2 h (T_w - T_\infty) \quad (2.72)$$

$$T_w = T_\infty + \frac{\dot{q} R}{3h} \quad (2.73)$$

The equation (2.52) can be written as:

$$T - T_\infty = \frac{\dot{q}}{6\lambda}(R^2 - r^2) + \frac{R\dot{q}}{3h} \;\; \ldots. \quad (2.74)$$

The equation (2.74) can be written as

$$T_{max} - T_\infty = \frac{\dot{q} R^2}{6\lambda} + \frac{\dot{q} R}{3h} \quad (2.75)$$

Equation 2.70 shows that the temperature distribution is parabolic. The rate of heat transfer at any cross-section can be determined using:

$$\phi = -\lambda \frac{dT}{dr} A \quad (2.76)$$

2.5 EXTENDED SURFACES (FINS) .

Extended surfaces, or fins, are used to improve heat transfer by increasing the surface area available for conduction and convection. They are widely used in applications requiring efficient heat dissipation, including heat exchangers, radiators, and electronic cooling systems.

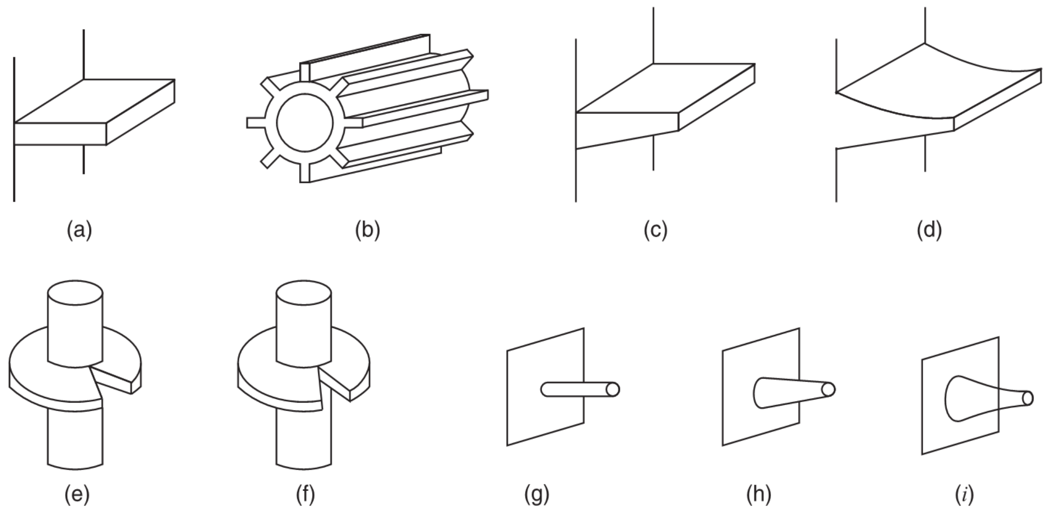


Figure 2.9. Schematic of different types of fins.

2.6 GENERAL THERMAL ANALYSIS

The heat transfer analysis of extended surfaces in the simplified configuration (figure 2.10) assumes a uniform cross-sectional area along the direction of heat transfer. A fin is used to increase the surface area for heat dissipation from a primary surface to the surrounding fluid. Heat is transferred by conduction from the base into the fin and is then dissipated to the surrounding fluid by convection from the fin's exposed surface. Under steady-state conditions, the energy balance for the fin is expressed as :

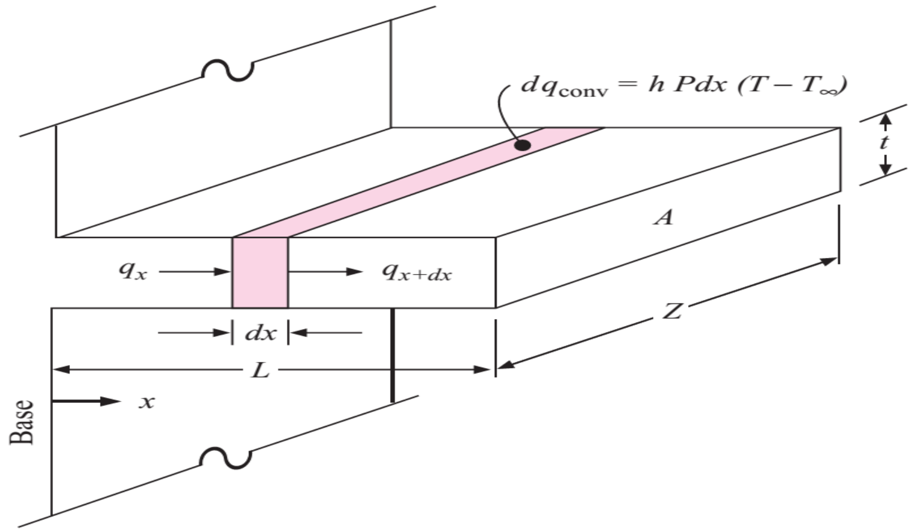


Figure. 2.10. Pin fin

Heat conducted into the fin at its base- Heat convected from the fin surface up to a specific section x=Heat conducted out of that section x into the rest of the fin material.

The primary quantities we need to determine are:

(i) the temperature distribution

(ii) the total heat transfer rate.

This process shows that both temperature and heat flux change continuously along the length of an extended surface (fin). The main goal of analyzing a fin is to determine two primary quantities: Temperature distribution along the length and total heat transfer rate.

The parameters taken into account in the analysis are : T_∞ the fluid temperature, T_b the base temperature, λ thermal conductivity of the material, (which is considered as constant), h Convective heat transfer coefficient, A the sectional area perpendicular to the heat flow direction, P perimeter exposed to the fluid, direction of convection.

To analyze the temperature distribution of a fin, we consider a small, infinitesimal control volume of length dx at a distance x from the base as illustrated in Figure 2.6. Applying the principle of energy balance for steady-state conditions, we find that the rate of energy entering the control volume must equal the rate of energy leaving it.

Rate of heat conduction into the element at section x - the rate of heat conduction out of the element at section x + dx -the rate of heat convection from the element's surface = 0.

$$-\lambda A\frac{dT}{dx}-\left[\lambda A\frac{dT}{dx}+\frac{d}{dx}\left(-\lambda A\frac{dT}{dx}\right)dx\right]-hPdx\left(T-T_\infty\right)=0 \tag{2.77}$$

We assume that the thermal conductivity (λ) and cross-sectional area (A) remain constant, and that the convective surface area of the element is P.dx .

$$\frac{d^2T}{dx^2}-\frac{hP}{\lambda A}\left(T-T_\infty\right)=0 \tag{2.78}$$

In order to solve the equation, two new variables θ and m are introduced.

$$\theta=\left(T-T_\infty\right) \text{ So, } \frac{d^2T}{dx^2}=\frac{d^2\theta}{dx^2} \tag{2.79}$$

$$m=\sqrt{\frac{hP}{\lambda A}} \tag{2.80}$$

The equation reduces to $$\frac{d^2\theta}{dx^2}-m^2\theta=0 \tag{2.81}$$

The general solution for this equation is

$$\theta=C_1e^{mx}+C_2e^{-mx} \tag{2.82}$$

The values of the constants C_1 and C_2 are determined by the applied boundary conditions. There are four distinct sets of boundary conditions we can consider, and each set yields a unique pair of values for C_1 and C_2.

Case 1: Long fin configuration:

The boundary conditions are:

$x \to \infty, \theta = 0$ and for $x = 0, \theta = T_b - T_\infty$

Thus $\theta = C_1 e^{mx} + C_2 e^{-mx}$ (2.83)

From first boundary condition, $C_1 = 0$, otherwise θ will become infinite which is not possible.

$\theta = C_2 e^{-mx}$

$x = 0, \theta = T_b - T_\infty$

$T_b - T_\infty = C_2 e^{-m0} = C_2$ so $C_2 = T_b - T_\infty = \theta_0$ (2.84)

$$\frac{\theta}{\theta_0} = \frac{T - T_\infty}{T_b - T_\infty} = e^{-mx}$$ (2.85

In this case the variation of temperature is exponential.

Case 2.Short fin end insulated:

At $x = 0, \theta = \theta_0 = T_b - T_\infty$, At $x = L, \frac{d\theta}{dx} = 0$ as the surface is insulated.

From the first condition $\theta = \left(C_1 e^{mx} + C_2 e^{-mx}\right)$ leads to

$$\theta_0 = C_1 + C_2 \tag{2.86.a}$$

$$\left.\frac{d\theta}{dx}\right|_L = m\left(-C_1 e^{mL} \ + \ C_2 e^{-mL}\right) = 0 \tag{2.86.b}$$

$$C_1 e^{mL} \ = C_2 e^{-mL} \text{ Or } C_2 \ = C_1 e^{2mL} \tag{2.86.c}$$

Using equations (2.70.a)and (2.70.c) $\theta_0 = C_1 + C_1 e^{2mL}$ Or $C_1 = \dfrac{\theta_0}{1+e^{2mL}}$ (2.87.a)

Using equation (2.70.c) and (2.71.a) $C_2 = \dfrac{\theta_0}{1+e^{2mL}} e^{2mL} = \dfrac{\theta_0}{1+e^{-2mL}}$ (2.87.b)

$$\begin{aligned} \frac{\theta}{\theta_0} &= \frac{e^{mx}}{1+e^{2mL}} + \frac{e^{-mx}}{1+e^{-2mL}} \\ &= \frac{e^{mx}e^{-mL}}{e^{mL}+e^{-mL}} + \frac{e^{-mx}e^{mL}}{e^{mL}+e^{-mL}} = \frac{e^{-m(L-x)}e^{m(L-x)}}{e^{mL}+e^{-mL}} \end{aligned} \tag{2.88}$$

$$\frac{T-T_\infty}{T_b-T_\infty} = \frac{\cosh\left[m\left(L-x\right)\right]}{\cosh\left(mL\right)} \tag{2.89}$$

Here, we are neglecting heat loss by convection from the fin tip. To mitigate the resulting error, the fin length can be effectively increased by $\Delta L = e/2$ Δ, where e is the fin's thickness. For fins with a circular cross-section, this correction becomes $\Delta L = D/4$.

The temperature ratio at the extremity is :

$$\frac{T_L-T_\infty}{T_0-T_\infty} = \frac{1}{\cosh\left(mL\right)} \tag{2.90}$$

<u>Case 3. Short fin with convection, h_L at the tip.</u>

The boundary conditions are

At $x=0, \theta=\theta_0$, At $x=L, -\lambda\frac{dT}{dx}=h_L\left(T_L-T_\infty\right)$

The calculation is more complex. The resulting equation is:

$$\frac{T-T_\infty}{T_b-T_\infty}=\frac{\cosh\left(m\left(L-x\right)\right)+\frac{h_L}{m.\lambda}\sinh\left(m\left(L-x\right)\right)}{\cosh\left(mL\right)+\frac{h_L}{m.\lambda}\sinh\left(mL\right)} \tag{2.91}$$

At the tip, the temperature ratio is:

$$\frac{T_L-T_\infty}{T_b-T_\infty}=\frac{1}{\cosh\left(mL\right)+\frac{h_L}{m.\lambda}\sinh\left(mL\right)} \tag{2.92}$$

Case 4 Specified end temperatures

At $x=0, \theta=T_{b1}-T_\infty$ where T_{b1} is the temperature at end 1

At $x=L, \theta=T_{b2}-T_\infty$ where T_{b2} is the temperature at end 2

In this case,.the resulting solution is :

$$\frac{T-T_\infty}{T_{b1}-T_\infty}=\frac{\left[\left(T_{b2}-T_\infty\right)/\left(T_{b1}-T_\infty\right)\right]\sinh\left(mx\right)+\sinh\left(m\left(L-x\right)\right)}{\sinh\left(mL\right)} \tag{2.93}$$

Table 2.1 presents these boundary conditions along with the resulting temperature distribution for each case.

Table 2.1 Temperature distribution in constant area fins for different boundary conditions $m = \sqrt{hp / \lambda A}$.

Boundary condition and general nomenclature	Temperature distribution
1.Long fin $x = 0, \theta = T_0 - T_\infty$	$\left(T - T_\infty\right) / \left(T_b - T_\infty\right) = e^{-mx}$
2.Short fin end insulated $x = 0, \theta = T_b - T_\infty$ $x = L, \dfrac{d\theta}{dx} = 0$	$\dfrac{T - T_\infty}{T_b - T_\infty} = \dfrac{\cosh\left[m\left(L - x\right)\right]}{\cosh\left(mL\right)}$
3.Short fin (convection at the tip h, considered) $x = 0, \theta = T_b - T_\infty$ $x = L, -\lambda \left.\dfrac{dT}{dx}\right\|_L = h_L\left(T - T_\infty\right)$	$\dfrac{T - T_\infty}{T_b - T_\infty} = \dfrac{\cosh\left(m\left(L - x\right)\right) + \dfrac{h_L}{m.\lambda}\sinh\left(m\left(L - x\right)\right)}{\cosh\left(mL\right) + \dfrac{h_L}{m.\lambda}\sinh\left(mL\right)}$
4.Fixed end temperature $x = 0, \theta = T_{01} - T_\infty$ $x = 0, \theta = T_{02} - T_\infty$	$\dfrac{T - T_\infty}{T_{b1} - T_\infty} = \dfrac{\left[\left(T_{b2} - T_\infty\right) / \left(T_{b1} - T_\infty\right)\right]\sinh\left(mx\right) + \sinh\left(m\left(L - x\right)\right)}{\sinh\left(mL\right)}$

2.7. PERFORMANCE OF FINS

Heat Transfer Rate from a Fin

The general expression for heat transfer from a fin depends significantly on the tip boundary condition. For the common case where the fin tip is insulated (i.e., no heat loss from the tip), or if the fin is long enough that the tip temperature is nearly equal to the ambient temperature T_∞, the heat transfer from the fin is given by .

$$Q_{fin} = \sqrt{h\lambda P.A}\left(T_b - T_\infty\right)\tanh\left(mL\right) \tag{2.94}$$

For an infinitely long fin:

$$Q_{fin} = \sqrt{h\lambda P.A}\left(T_b - T_\infty\right) \tag{2.95}$$

Fin Effectiveness

Fin effectiveness ε_{fin} measures a fin's ability to enhance heat transfer by comparing the heat transferred from the fin to the heat that would be transferred from the same area if there were no fin.

$$\varepsilon_{fin} = \frac{\text{Actual heat transfer rate from the fin}}{\text{Heat transfer rate from the base area if no fin was present}} \tag{2.96}$$

Fin Efficiency

Fin efficiency is defined as the ratio of the actual heat transferred by a fin to the maximum possible heat transfer if the entire fin surface were at the base temperature (T_b). Mathematically, it is expressed as :

$$\eta_{fin} = \frac{\text{Actual heat transfer rate from the fin}}{\text{Ideal heat transfer rate if entire fin was at } T_b} \tag{2.97}$$

Therefore

$$\eta_{fin} = \frac{Q_{fin}}{h.A_{fin}\ \left(T_b - T_\infty\right)} \tag{2.98}$$

EXERCISES

EXERCISE 2.1

A boiler furnace with vertical walls measuring 4m x3m (total height 3m) is constructed with a three-layer composite wall. The wall layers, starting from the interior, are:

-Fire Brick: Thickness L_1 of 25 cm and thermal conductivity λ_1 of 0.4 [W/(m. K)]

-Ceramic Blanket Insulation: L_2 of 8 cm and thermal conductivity λ_2 of 0.2 [W/(m. K)]

-Steel Protective Layer: L_3 of 2 mm and thermal conductivity λ_3 of 55 [W/ (m. K)].

The temperature of the interior surface of the fire brick is measured at 600°C, and the temperature at the outer surface of the ceramic blanket insulation is 60°C.

1°/Determine the total heat loss rate through the vertical walls of the furnace.

2°/Determine the temperature at the interface between the fire brick and the ceramic blanket insulation.

Data: Area of rectangular wall is $l.b = 12\left[m^2\right]$,

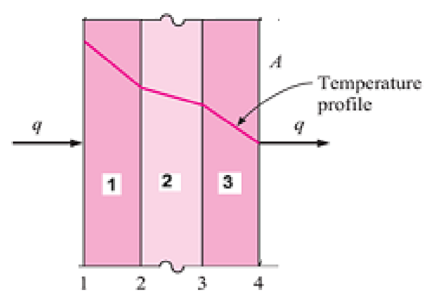
$l = 4[m], b = 3[m], h = 3[m]$

For Fire brick $\begin{cases} L_1 = 25[cm] \\ \lambda_1 = 0,4\left[W/(m.K)\right] \end{cases}$,

for Steel $\begin{cases} L_2 = 0,2[cm] \\ \lambda_2 = 54\left[W/(m.K)\right] \end{cases}$

For insulation $\begin{cases} L_3 = 8[cm] \\ \lambda_3 = 0,2\left[W/(m.K)\right] \end{cases}$

$T_1 = 600[°C]$ and $T_2 = 60[°C]$



1°/Determination of heat transfer rate

We know that, $\phi = \dfrac{\Delta T}{R_{th-eq}}$

Where: $\Delta T = T_1 - T_4$

And $R_{th-eq} = R_{th1} + R_{th2} + R_{th3}$

$$R_{th1} = \frac{L_1}{\lambda_1 A} = \frac{0,25}{0,4\times 12} = 0.0521\left[°C/W\right]$$

$$R_{th2} = \frac{L_2}{\lambda_2 A} = \frac{0,08}{0,2\times 12} = 0.0333\left[°C/W\right]$$

$$R_{th3} = \frac{L_3}{\lambda_3 A} = \frac{0,002}{54\times 12} = 0.0000031\left[°C/W\right]$$

$$\phi = \frac{T_1 - T_4}{R_{th1} + R_{th2} + R_{th3}}$$

$$\phi = \frac{600-60}{0,0521 + 0,0000031 + 0,0333}$$

So $\phi = 6320,96\left[W\right]$

2°/ Determination of the temperature drop across the steel layer (T_3-T_4)

$$\phi = \frac{T_3 - T_4}{R_{th3}}$$

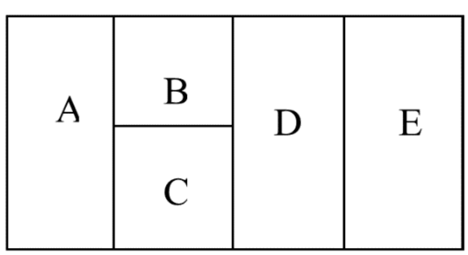
$$T_3 - T_4 = \phi . R_{th3}$$

=6320,96(0,0000031)

Thus $T_3 - T_4$ =0,0196°C

EXERCISE 2.2

A composite wall, detailed in the figure below, is constructed of five materials (A, B, C, D, and E). Sections B and C are arranged in parallel, while sections A, the B-C combination, D, and E are in series.



The thermal conductivities and cross-sectional areas for each section are given in the flowing table:

Section	A	B	C	D	E
Thermal conductivity $[W/(m.K)]$	50	10	6,67	20	30
Area (m^2)	1	0,5	0,5	1	1

If the temperature entering at wall A is 800°C and leaving at wall E is 100 °C.

1.Calculate the heat transfer through the composite wall.

Solution

$T_i = 800°C$ and $T_o = 100°C$

Solution

We know that:

$$Q = \frac{(\Delta T)_{overall}}{\sum R_{th}}$$

$$R_{th1} = R_{thA} = \frac{L_A}{\lambda_A A_A}$$

The walls B and C are in parallel Then, $\frac{1}{R_{th2}} = \frac{1}{R_{thB}} + \frac{1}{R_{thC}} = \frac{R_{thB} + R_{thC}}{R_{thB} R_{thC}}$

$$R_{th2} = \frac{R_{thB} R_{thC}}{R_{thB} + R_{thC}}, \text{Where } R_{thB} = \frac{L_B}{\lambda_B A_B} \ \ and \ \ R_{thC} = \frac{L_C}{\lambda_C A_C}$$

$$R_{th3} = R_{thD} = \frac{L_D}{\lambda_D A_D}$$

$$R_{th4} = R_{thE} = \frac{L_E}{\lambda_E A_E}$$

$$R_{th1} = R_{thA} = \frac{1}{50 \times 1} = 0,02 (K/W)$$

$$R_{thB}=\frac{1}{10\times 0,5}=0,2\left[°C/W\right] \text{ and } R_{thC}=\frac{1}{6,67\times 0,5}=0,2969\left[°C/W\right]$$

$$R_{th2}=\frac{R_{thB}R_{thC}}{R_{thB}+R_{thC}}=\frac{0,2\times 0,299}{0,2+0,299}=0,1198\left[°C/W\right]$$

$$R_{th3}=R_{thD}=\frac{L_D}{\lambda_D A_D}=\frac{1}{20\times 1}=0,05\left[°C/W\right]$$

$$R_{th4}=R_{thE}=\frac{L_E}{\lambda_E A_E}=\frac{1}{30\times 1}=0,0333\left[°C/W\right]$$

Hence

$$\phi=\frac{T_i-T_o}{\sum_{i=1}^{4}R_{th}}=\frac{800-100}{0,02+0,1198+0,05+0,0333}=3137,61\left[W\right]$$

<u>EXERCISE 2.3</u>

A hollow cylindrical wall is composed of a material with a thermal conductivity of 70 W/m.K This cylinder has an inner diameter D_1 of 5 cm and an outer diameter D_2 of 10 cm. The inner surface is maintained at a uniform temperature T_1=300K and the outer surface is maintained at a temperature T_2 of 100

1.Determine the temperature T_m at the radial location midway between the inner and outer surfaces.

2.Determine the rate of heat transfer per unit length through the cylinder.

Solution

We know that : $\phi=\dfrac{T_1-T_2}{\ln\left(\dfrac{r_2}{r_1}\right)}2\pi\lambda L$

Hence

$$\phi = \frac{6,28.(1).(70)(300-100)}{\ln\left(\frac{5}{2,5}\right)} = 126868.6[W] = 126.86\ [kW]$$

At half way between R_1 and R_2 radius, $R_m = \frac{2,5+5}{2} = 3,75[cm]$ since ϕ remains the same under steady state conditions

$$\phi = \frac{(T_1 - T_m)}{\ln\left(\frac{R_m}{R_1}\right)} 2\pi\lambda L = \frac{(T_1 - T_2)}{\ln\left(\frac{R_2}{R_1}\right)} 2\pi\lambda L$$

$$\frac{(T_1 - T_m)}{\ln\left(\frac{R_m}{R_1}\right)} = \frac{(T_1 - T_2)}{\ln\left(\frac{R_2}{R_1}\right)}$$

$$(T_1 - T_m) = (T_1 - T_2)\frac{\ln\left(\frac{R_m}{R_1}\right)}{\ln\left(\frac{R_2}{R_1}\right)} \Rightarrow (T_1 - T_m) = 200\frac{\ln\left(\frac{3,75}{2,5}\right)}{\ln\left(\frac{5}{2,5}\right)} = 117[°C]$$

Therefore

$$T_m = T_1 - 117 = 183[°C]$$

EXERCISE 2.4

A steel pipe transports steam at an internal temperature of 260 °C. The pipe has an inner diameter D_1 of 100 mm and a wall thickness e_1 of 7 mm.

The pipe is insulated by two concentric layers:

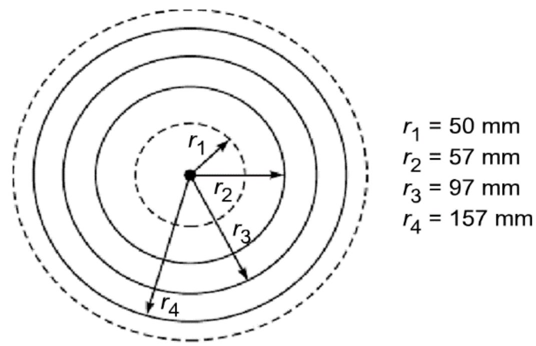
1. A primary layer of glass wool with a thickness e_2 of 40 mm.
2. An outer layer of asbestos felt with a thickness e_3 of 60 mm.

The surrounding ambient air temperature T_∞ is 20 °C.

Thermal Properties are given in the following table

Parameter	Symbol	Value	Unit
Inner Convective Heat Transfer Coefficient	h_{in}	550	W/m²·K
Outer Convective Heat Transfer Coefficient	h_{out}	15	W/m²·K
Thermal Conductivity of Steel	λ_{steel}	50	W/m·K
Thermal Conductivity of Glass Wool	λ_{glass}	0.09	W/m·K
Thermal Conductivity of Asbestos Felt	$\lambda_{asbestos}$	0.07	W/m·K

Calculate:

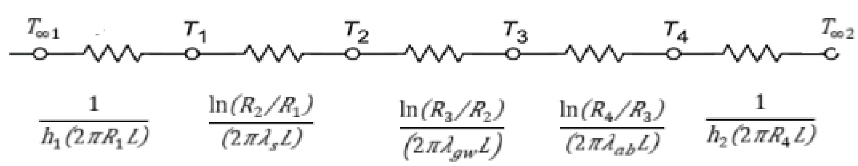


1/Rate of heat loss per unit length of pipe.

2/Temperature at each x section of the pipe.

Solution

Equivalent electrical circuit is given by :



$$\phi = \frac{T_{\infty 1} - T_{\infty 2}}{\frac{1}{h_1\left(2\pi R_1 L\right)} + \frac{\ln\left(R_2 / R_1\right)}{\left(2\pi\lambda_s L\right)} + \frac{\ln\left(R_3 / R_2\right)}{\left(2\pi\lambda_{gw} L\right)} + \frac{\ln\left(R_4 / R_3\right)}{\left(2\pi\lambda_{ab} L\right)} + \frac{1}{h_2\left(2\pi R_4 L\right)}} =$$

$$= \frac{\left(260-20\right)2\pi \times 1}{\frac{1000}{550\left(50\right)} + \frac{\ln\left(57/50\right)}{\left(50\right)} + \frac{\ln\left(97/57\right)}{\left(0,09\right)} + \frac{\ln\left(157/97\right)}{\left(0,07\right)} + \frac{1000}{15\left(157\right)}}$$

$$= 50,50\left[W\right]$$

$$\phi = \frac{T_{\infty 1} - T_1}{\frac{1}{h_1\left(2\pi R_1 L\right)}}$$

$$50,5 = \frac{260 - T_1}{0,064}$$

Hence $T_1 = -50,5\left(0,064\right) + 260 = 256,77°C$

Now, we have $\phi = \frac{T_1 - T_2}{\frac{\ln\left(R_2 / R_1\right)}{\left(2\pi\lambda_s L\right)}} \Rightarrow T_2 = T_1 - \phi\frac{\ln\left(R_2 / R_1\right)}{\left(2\pi\lambda_s L\right)}$

Thus $T_2 = 256,77 - 50.5\frac{\ln\left(57/50\right)}{\left(100\pi\right)} = 256.74°C$

Now, we have for next junction:

$$\phi = \frac{T_2 - T_3}{\frac{\ln\left(R_3 / R_2\right)}{\left(2\pi\lambda_{gw} L\right)}} \Rightarrow T_3 = T_2 - \phi\frac{\ln\left(R_3 / R_2\right)}{\left(2\pi\lambda_{gw} L\right)}$$

$$T_3 = 256,76 - 50.5\frac{\ln\left(97/57\right)}{\left(0,18\pi\right)} = 209,24\left[°C\right]$$

Now, we have for next junction

$$\phi = \frac{T_3 - T_4}{\frac{\ln\left(R_4 / R_3\right)}{\left(2\pi\lambda_{ab} L\right)}} \Rightarrow T_4 = T_3 - \phi \frac{\ln\left(R_4 / R_3\right)}{\left(2\pi\lambda_{ab} L\right)}$$

$$T_4 = 209,29 - 50.5 \frac{\ln\left(157/97\right)}{\left(0,14\pi\right)} = 153,92\left[°C\right]$$

EXERCISE 2.5

A hollow copper conductor (D_1=13mm and D_2=50mm) carries a current density of 5000 A/cm^2. The outer surface is fixed at 40°C, and the inner surface is adiabatic.

Data: ρ_e = 2.10^{-6} Ω.cm and λ=381 [W/m.K]

1.Find the location and magnitude of the maximum internal temperature.

Solution

Heat generation rate per unit volume,

$$\dot{q} = \frac{I^2 R}{A.L} = \frac{\left(5000.10^4.A\right)^2}{A.L}.\frac{\rho L}{A} = \left(5000.10^4\right)^2.2.10^{-8} = 50.10^6\left[W/m^3\right]$$

The differential equation for one-dimensional heatflow with heat generation in cylindrical coordinates $\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{\dot{q}.r}{\lambda} = 0$ is

Integration of this equation gives

$$r\frac{dT}{dr} + \frac{\dot{q}.r^2}{2\lambda} + C_1 = 0 \text{ so } \frac{dT}{dr} + \frac{\dot{q}.r}{2\lambda} + \frac{C_1}{r} = 0$$

Further integration gives the temperature distribution equation as

$$T+\frac{\dot{q}.r^2}{4\lambda}+C_1\ln r+C_2=0 \qquad (*)$$

The constants of the integration C_1 and C_2 are determined from the boundary conditions, which are:

For r=R_1 , $\frac{dT}{dr}=0$ as no heat is removed from the inner surface.

For r=R_2 , $T=T_2$

The first boundary condition gives

$$C_1=\frac{-\dot{q}.R_1^{\ 2}}{2\lambda}$$

Inserting value of constant C_1 in the temperature distribution equation, we get

$$T+\frac{\dot{q}.r^2}{4\lambda}+\frac{-\dot{q}.R_1^{\ 2}}{2\lambda}\ln r+C_2=0$$

The second boundary condition give

$$T_2+\frac{\dot{q}.R_2^{\ 2}}{4\lambda}-\frac{\dot{q}.R_1^{\ 2}}{2\lambda}\ln R_2+C_2=0 \text{ Or } C_2=\frac{\dot{q}.R_1^{\ 2}}{2\lambda}\ln R_2-T_2-\frac{\dot{q}.R_2^{\ 2}}{4\lambda}$$

N.A: $C_1=\frac{-\dot{q}.R_1^{\ 2}}{2\lambda}=\frac{-50.10^6}{2(381)}\left(6,5.10^{-3}\right)^2=-2.772\left[°C/m\right]$

$$C_2=\frac{\dot{q}.R_1^{\ 2}}{2\lambda}\ln R_2-T_2-\frac{\dot{q}.R_2^{\ 2}}{4\lambda}=-70.82\left[°C\right]$$

Substitution of the constant values and other parameters in equation (*) gives

$$T + 0.0328r^2 - 2.772\ln r - 70.82 = 0$$

The maximum temperature is at the inner surface (r= 6,5.10^{-3} m) and is

$$T = -0.0328\left(6,5.10^{-3}\right)^2 + 2.772\ln\left(6,5.10^{-3}\right) + 70.82 = 55,47°C$$

The heat transfer at the outer surface is

$$\phi = -\lambda \left\| \overrightarrow{gradT} \right\|_{r=R_2} .A$$

Therefore $\phi = 91542\left[W\right]$

EXERCISE 2.6

A hollow spherical shell is constructed from a material with a thermal conductivity λ of 30 (W/m.K). The sphere has an inner diameter D_1 of 12 cm and an outer diameter D_2 of 21cm. Heat is generated uniformly within the solid material at a rate ($ of 5.10^6(W/m^3) . The inner surface is perfectly insulated (adiabatic), and the outer surface temperature is maintained at a constant temperature of 360°C

1.Determine the maximum temperature T_{max} achieved within the spherical shell under steady-state conditions.

Solution

$$\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) + \frac{\dot{q}.r^2}{\lambda} = 0$$

Integrating this equation twice gives:

$$T(r) = -\frac{\dot{q}.r^2}{6\lambda} - \frac{C_1}{r} + C_2$$

At r=R_1,the heat flux is zero, therefore . $\frac{dT}{dr} = 0$

$-\frac{\dot{q}.R_1}{3\lambda} + \frac{C_1}{R_1^{\ 2}} = 0$ Hence $C_1 = \frac{\dot{q}.R_1^{\ 3}}{3\lambda}$

At r =R_2, T(r=R_2) = T2 = 360 °Therefore

$$C_2 = T_2 + \frac{\dot{q}.R_2^{\ 2}}{6\lambda} + \frac{\dot{q}.R_1^{\ 3}}{3\lambda R_2}$$

So, the temperature distribution is

$$T(r) = -\frac{\dot{q}.r^2}{6\lambda} - \frac{\dot{q}.R_1^{\ 3}}{3\lambda r} + T_2 + \frac{\dot{q}.R_2^{\ 2}}{6\lambda} + \frac{\dot{q}.R_1^{\ 3}}{3\lambda R_2}$$

The maximum temperature is reached at (r= R_1=6.10^{-2} m)

$$T_{\max} = T(r = R_1) = 480.54°C$$

EXERCISE 2.7

A long cylindrical shaft (60 mm diameter) acts as an extended surface (fin) dissipating heat generated by an adjacent bearing. The temperature at the base of the shaft (the shaft end near the bearing) is maintained at 60°C above the ambient temperature T_{∞}. The heat is transferred from the shaft surface to the surrounding air via convection, with a heat transfer coefficient of 7 [W/m^2.K].

The shaft material has a thermal conductivity of 60 W/(m.K).Assuming the shaft can be modeled as an infinitely long fin,

1.Determine the mathematical expression for the temperature distribution along the shaft's axis.

2. Determine the total rate of heat transferred from the shaft to the ambient air.

Solution

The temperature distribution is given by:

$\frac{T-T_\infty}{T_s-T_\infty}=e^{-mx}$ Here $m=\sqrt{\frac{hP}{\lambda A_C}}$ so $m=\sqrt{\frac{h\pi D}{\lambda\frac{\pi D^2}{4}}}=\sqrt{\frac{4h}{\lambda D}}$ A.N $m=\sqrt{\frac{4\times 7}{60(0,06)}}=2,79$ and

$T_s-T_\infty=60^\circ C$ Hence $T-T_\infty=60e^{-2,79x}$

The rate of heat rejected from the entire surface area of the shaft is :

$$\phi_{\text{fin}}=\sqrt{hP\lambda A_c}\left(T_s-T_\infty\right)$$

Or $\phi_{\text{fin}}=\sqrt{7\times\pi\times 0,06\times 60\times(\pi/4)\times(0,06)^2}\times 60=28,39[W]$.

CHAPTER 3 TWO-DIMENSIONAL STATIONARY CONDUCTION

3.1 INTRODUCTION

Two-dimensional heat conduction involves temperature variation in two spatial directions and is governed by the 2D heat equation. Analytical solutions are possible for simple geometries and boundary conditions using methods like separation of variables. However, complex domains often require numerical approaches such as finite difference or finite element methods.

3.2 ANALYTICAL SOLUTION OF TWO-DIMENSIONAL HEAT CONDUCTION PROBLEMS.

Consider a long rectangular bar (infinite in the z-direction), as illustrated in Figure 3.1. Three of its lateral sides are maintained at a constant temperature T_o. The temperature along the fourth side (y=H) is given by an arbitrary function f(x).

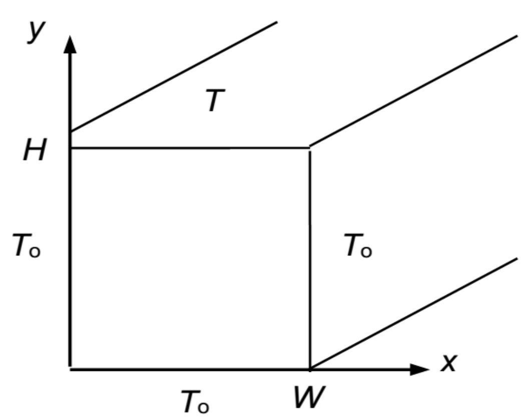


Figure 3.1 A rectangular section bar with given thermal boundary conditions .

We know that the equation of heat transfer in steady state without heat generation is expressed by:

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0 \tag{3.1}$$

Using $\theta = T - T_0$, the Laplace equation (3.1), is transformed to:

$$\frac{\partial^2 \theta}{\partial x^2}+\frac{\partial^2 \theta}{\partial y^2}=0 \tag{3.2}$$

The boundary conditions are :

$$\begin{aligned}&\text{At x} = 0,\ \theta = 0, \text{At y} = 0,\ \theta = 0,\\ &\text{At x} = W,\ \theta = 0, \text{At y} = H,\ \theta = f(x),\end{aligned} \tag{3.3}$$

The solution of equation (3.2) is obtained by using the separation of variables method, which relies on the assumption that the solution can be expressed as a product of functions. This yields:

$$\theta(x,y) = X.Y \tag{3.4}$$

Where X=X(x) and Y=Y(y).

Substitution in equation (3.2) gives

$$-\frac{1}{X}\frac{d^2X}{dx^2}=\frac{1}{Y}\frac{d^2Y}{dy^2} \tag{3.5}$$

Because x and y are independent variables, the left-hand side and the right-hand side of equation (3.5) are independent of each other. Consequently, for the equation to be true, both sides must be equal to a common constant. Let's denote this positive constant as K^2. This gives us:

$$-\frac{1}{X}\frac{d^2X}{dx^2}=\frac{1}{Y}\frac{d^2Y}{dy^2}=K^2 \tag{3.6}$$

Thus, we get two ordinary differential equations as:

$$\frac{d^2X}{dx^2}+K^2X=0 \tag{3.7}$$

$$\frac{d^2Y}{dy^2}-K^2Y=0 \tag{3.8}$$

The value of the constant K is to be determined from the given boundary conditions.

The general solution of equation (3.7) is

$$X=\left[C_1\cos(Kx)+C_2\sin(Kx)\right] \tag{3.9}$$

and that of equation (3.8) is:

$$Y=\left[C_3\exp(-Ky)+C_4\exp(Ky)\right] \tag{3.10}$$

Substitution in equation (3.4) gives [20].

$$\theta(x,y)=XY=\left[C_1\cos(Kx)+C_2\sin(Kx)\right]\left[C_3\exp(-Ky)+C_4\exp(Ky)\right] \tag{3.11}$$

Applying the boundary conditions, we find :

At y=0, $$\left[C_1\cos(Kx)+C_2\sin(Kx)\right](C_3+C_4)=0 \tag{3.12.a}$$

At x=0, $$C_1\left[C_3\exp(-Ky)+C_4\exp(Ky)\right]=0 \tag{3.12.b}$$

At x=W, $$\left[C_1\cos(K.W)+C_2\sin(K.W)\right]\left[C_3\exp(-Ky)+C_4\exp(Ky)\right]=0 \tag{3.12.c}$$

Equations (3.12.a) and (3.12.b) give

$C_3 = -C_4$ and $C_1 = 0$

Substitution in equation (3.12.c) gives:

$$C_2 C_4 \sin(KW)\left[\exp(Ky) - \exp(-Ky)\right] = 0 \tag{3.13}$$

Equation (3.13) requires that:

$$\sin(KW) = 0 \tag{3.14}$$

Therefore, $K = \dfrac{\pi n}{W}$, where n is a positive integer.

By substituting values of constants C_1 to C_4 and K in equation (3.11), we get .

$$\theta(x, y) = T - T_0 = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{W}\right)\sinh\left(\frac{n\pi y}{W}\right) \tag{3.15}$$

In this form, the exponential term has been replaced by sinh (nπy/W), and the constants C_2 and C_4 have been combined. The complete solution to the differential equation is obtained by summing the individual solutions for each value of n, extending to infinity.

The fourth boundary condition gives

$$f(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{W}\right)\sinh\left(\frac{n\pi H}{W}\right) \tag{3.16}$$

The terms $C_n \sinh\left(\dfrac{n\pi H}{W}\right)$ are identified as the coefficients of the Fourier sine series for the function f(x) in the interval 0<x<L. In other words:

$$C_n \sinh\left(\frac{n\pi H}{W}\right) = \frac{2}{W}\int_0^W f\left(x\right)\sin\left(\frac{n\pi x}{W}\right)dx \tag{3.17}$$

and thus,

$$\theta\left(x,y\right) = T - T_0 = \frac{2}{W}\sum_{n=1}^{\infty}\left[\frac{1}{\sinh\left(\frac{n\pi H}{W}\right)}\int_0^W f\left(x\right)\sin\left(\frac{n\pi x}{W}\right)dx\right]\sin\left(\frac{n\pi x}{W}\right)\sinh\left(\frac{n\pi y}{W}\right) \tag{3.18}$$

The temperature distribution is represented in figure 3.2. In this figure, the heat flow lines (shown as dashed lines) are always perpendicular to the lines of constant temperature, known as isotherms (shown as solid lines).

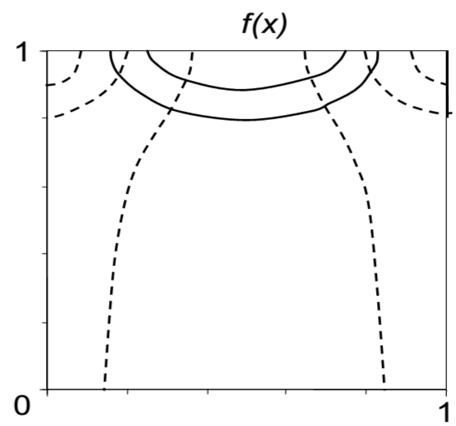


Figure. 3.2 Isotherms and heat flow lines in a rectangular plate.

3.3 FINITE DIFFERENCE ANALYSIS OF CONDUCTION

In modern heat transfer analysis, numerical methods like the finite difference method and finite element method are the most common approaches for solving complex conduction problems. We'll now introduce the basic concepts of the finite difference method and its application to heat conduction.

3.4 ENERGY BALANCE METHOD.

The energy balance method is an alternative and often more intuitive approach to developing finite-difference equations, especially for problems involving multiple materials, internal heat generation, or complex boundary geometries.

The key of the finite difference method is applying the conservation of energy principle to a control volume surrounding each nodal region to derive its finite-difference equation. To handle the uncertainty of heat flow direction, we assume, for the purpose of formulation, that all heat transfer is into the control volume associated with a given node (m, n).

For steady-state problems with heat generation, the energy balance applied is :

$$E_g + E_{in} = 0 \tag{3.19}$$

We consider two-dimensional steady-state heat conduction problem, the energy balance for a control volume enclosing an interior node (m, n) as shown in figure 4.5, is based on the principle of energy conservation. In this case, energy is exchanged through conduction with the four adjacent nodes (left, right, top, and bottom) and is balanced by internal heat generation. Consequently:

$$\sum_{i=1}^{4} \phi_{i \rightarrow (m,n)} + \dot{q}\left(\Delta x . \Delta y . 1\right) = 0 \tag{3.20}$$

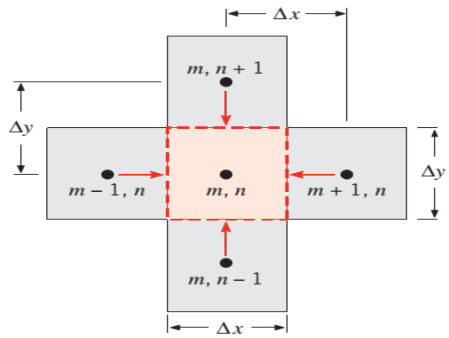


Figure 4.5 Conduction to an interior node from its adjoining nodes .

For a unit depth and under the assumption that heat conduction occurs only along paths parallel to the x and y axes, the heat conduction rate $\phi_{i \to (m,n)}$ between a neighboring node i and the central node (m, n) can be determined from Fourier's law.

The rate of heat transfer by conduction from the adjacent node (m−1, n) to node (m, n) is given by:

$$\phi_{(m-1,n) \to (m,n)} = \lambda (\Delta y.1) \frac{T_{m-1,n} - T_{m,n}}{\Delta x} \tag{3.21}$$

In this expression, the area for heat transfer is (Δy·1), and the term $\frac{T_{m-1,n} - T_{m,n}}{\Delta x}$ is the finite difference approximation of the temperature gradient, ∂T/∂x. The remaining conduction rates from the other adjacent nodes into the central node (m,n) can be formulated analogously [5]:

$$\begin{aligned} \phi_{(m+1,n) \to (m,n)} &= \lambda (\Delta y.1) \frac{T_{m+1,n} - T_{m,n}}{\Delta x} \\ \phi_{(m,n+1) \to (m,n)} &= \lambda (\Delta x.1) \frac{T_{m,n+1} - T_{m,n}}{\Delta y} \\ \phi_{(m,n-1) \to (m,n)} &= \lambda (\Delta x.1) \frac{T_{m,n-1} - T_{m,n}}{\Delta y} \end{aligned} \tag{3.22}$$

When using the finite difference method, heat transfer rates are systematically calculated by subtracting the temperature of the central node, $T_{m,n}$, from the temperature of its adjacent neighbor. This sign convention is a direct consequence of the initial assumption that all heat flow is directed into the control volume of node (m, n). By substituting the conduction rate expressions from the four adjacent nodes into the energy balance equation and assuming a uniform grid spacing $\Delta x = \Delta y$, the governing finite-difference equation for an interior node with heat generation becomes:

$$T_{m,n+1} + T_{m,n-1} + T_{m+1,n} + T_{m-1,n} + \frac{\dot{q}\left(\Delta x\right)^2}{\lambda} - 4T_{m,n} = 0 \qquad (3.23)$$

In the absence of internal heat generation $\dot{q} = 0$, the derived finite-difference expression for an interior node reduces to: $T_{m,n+1} + T_{m,n-1} + T_{m+1,n} + T_{m-1,n} - 4T_{m,n} = 0$

It's important to remember that a finite-difference equation is required for each node with an unknown temperature. For nodes located on insulated or convective surfaces, the finite-difference equation must be formulated specifically for that node by directly applying the energy balance method.

To provide a more detailed illustration of the energy balance method, let's analyze the specific case of the node located at the internal corner of the geometry shown in figure 4.6. This particular node represents the three-quarter shaded area shown and exchanges energy with a surrounding fluid at a temperature T_∞ through the convection process and with four neighboring nodes in the solid material through conduction. This transfer occurs along four distinct pathways.

Assuming steady-state conditions and uniform grid spacing, the heat transfer rates by conduction from the four neighboring nodes to the central node (m, n) can be expressed using simplified Fourier's law as follows:

$$\phi_{(m-1,n)\to(m,n)} = \lambda\left(\Delta y.1\right)\frac{T_{m-1,n} - T_{m,n}}{\Delta x} \tag{3.24}$$

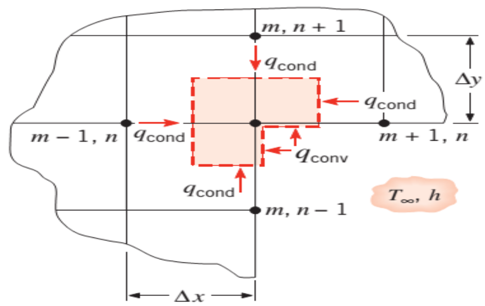


Figure4.6 : Formulation of the finite-difference equation for an internal corner of a solid with surface convection**.**

$$\begin{aligned}
\phi_{(m,n+1)\to(m,n)} &= \lambda\left(\Delta x.1\right)\frac{T_{m,n+1} - T_{m,n}}{\Delta y} \\
\phi_{(m+1,n)\to(m,n)} &= \lambda\left(\frac{\Delta y}{2}.1\right)\frac{T_{m+1,n} - T_{m,n}}{\Delta x} \\
\phi_{(m,n-1)\to(m,n)} &= \lambda\left(\frac{\Delta x}{2}.1\right)\frac{T_{m,n-1} - T_{m,n}}{\Delta y}
\end{aligned} \tag{3.25}$$

Notice that heat conduction from the full nodes, (m−1, n) and (m, n−1), occurs across the full areas of $\left(\Delta y.1\right)$ and $\left(\Delta x.1\right)$, respectively. However, the conduction paths from the partial nodes, (m+1, n) and (m, n-1), have widths of only $\left(\Delta y / 2\right)$ and $\left(\Delta x / 2\right)$, respectively. Furthermore, the node (m, n) also exchanges heat with the surrounding fluid via convection. This occurs over a surface area that is effectively a quarter of the full nodal area, or a half-length in both the x and y directions.

The total convective heat transfer rate ϕ_{conv} is given by :

$$\phi_{(\infty)\rightarrow(m,n)} = h\left(\frac{\Delta x}{2}.1\right)\left(T_\infty - T_{m,n}\right) + h\left(\frac{\Delta y}{2}.1\right)\left(T_\infty - T_{m,n}\right) \tag{3.26}$$

The previous formulation of the energy balance for a corner node relies on the key assumption that the exposed surfaces of the corner have a uniform temperature, which is equal to the temperature of the node itself, $T_{m,n}$

This approach is consistent with the fundamental concept of the finite difference method, where a single temperature value assigned to a node is an approximation. It represents the average temperature of the entire nodal region, rather than accounting for the actual, continuous temperature variations within that small area.

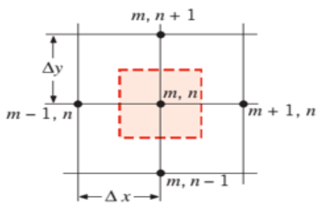
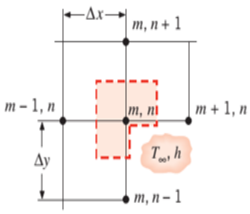
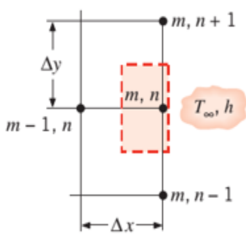
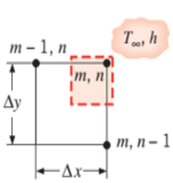
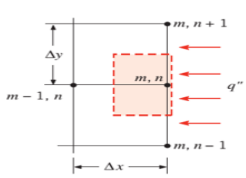
Based on the principles of energy conservation and the finite difference method under two-dimensional, steady-state conditions without internal heat generation, the sum of all heat transfer rates into a control volume must equal zero. By combining the heat conduction terms from the four adjacent nodes and the convection term from the surrounding fluid for the specific case of an internal corner node, and rearranging the equation, we arrive at the following finite-difference equation :

$$T_{m-1,n} + T_{m,n+1} + \frac{1}{2}\left(T_{m+1,n} + T_{m,n-1}\right) + \frac{h\Delta x}{\lambda}T_\infty - \left(3 + \frac{h\Delta x}{\lambda}\right)T_{m,n} = 0 \tag{3.27}$$

where the mesh is $\left(\Delta x = \Delta y\right)$.

Table 3.1 provides a summary of these equations for several common configurations, assuming no internal energy generation.

Table 3.1: nodal energy balance equations for different configurations.

Configuration	Finite difference equation for Δx=Δy
	$T_{m,n+1}+T_{m,n-1}+T_{m+1,n}+T_{m-1,n}-4T_{m,n}=0$
	$T_{m,n+1}+T_{m,n-1}+T_{m+1,n}+T_{m-1,n}+\frac{\dot{q}(\Delta x)^2}{\lambda}-4T_{m,n}=0$
	$\left(2T_{m-1,n}+T_{m,n+1}+T_{m,n-1}\right)+2\frac{h\Delta x}{\lambda}T_{\infty}-2\left(2+\frac{h\Delta x}{\lambda}\right)T_{m,n}=0$
	$\left(T_{m,n-1}+T_{m-1,n}\right)+2\frac{h\Delta x}{\lambda}T_{\infty}-2\left(1+\frac{h\Delta x}{\lambda}\right)T_{m,n}=0$
	$2T_{m-1,n}+T_{m,n+1}+T_{m,n-1}+2\frac{\dot{q}(\Delta x)}{\lambda}-4T_{m,n}=0$

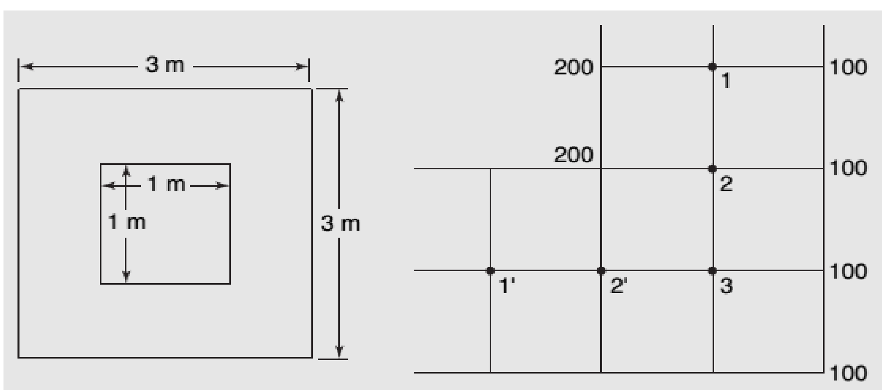
EXERCISES

EXERCISE 3.1

A hollow square duct of the configuration shown (left) has its surfaces maintained at 200 and 100 K, respectively. Determine the steady-state heat transfer rate between the hot and cold surfaces of this duct.The wall material has a thermal conductivity of 1,21W/m. K

We may take advantage of the eightfold symmetry of this figure to lay out the simple

The grid chosen is square with $\Delta x = \Delta y = 0,5m$: Three interior node points are thus identified; their temperatures may be determined by proper application of finite difference method



$$T_1 = \frac{200 + 100 + 2T_2}{4}$$

$$T_2 = \frac{200 + 100 + T_1 + T_3}{4}$$

$$T_3 = \frac{100 + 100 + 2T_2}{4}$$

This set of three equations and three unknowns may be solved quite easily to yield the following:

T_1=145.83K, T_2=141.67K and T_3=120.83K

The temperatures just obtained may now be used to find heat transfer. Implicit in the procedure of laying out a grid of the sort, we have specified is the assumption that heat flows in the x and y directions between nodes. On this basis heat transfer occurs from the hot surface to the interior only to nodes 1 and 2; heat transfer occurs to the cooler surface from nodes 1, 2, and 3.We should also recall that the section of duct that has been analysed is one-eight h of the total thus, of the heat transfer to and from node 1, only one half should be properly considered as part of the element analysed.

We now solve for the heat transfer rate from the hotter surface, and write

$$\begin{aligned}\varphi &= \lambda\frac{\left(200-T_1\right)}{2}+\lambda\left(200-T_2\right)\\ &= \lambda\left[\frac{\left(200-145.83\right)}{2}+\left(200-141{,}67\right)\right]\\ &= 85{,}415\lambda\end{aligned}$$

A similar accounting for the heat flow from nodes 1, 2, and 3 to the cooler surface is written as

$$\begin{aligned}\varphi &= \lambda\frac{\left(T_1-100\right)}{2}+\lambda\left(T_2-100\right)+\lambda\left(T_3-100\right)\\ &= \lambda\left[\frac{\left(145.83-100\right)}{2}+\left(141{,}67-100\right)+\left(120{,}83-100\right)\right]\\ &= 85{,}415\lambda\end{aligned}$$

Observe that these two different means of solving for q yield identical results. This is obviously a requirement of the analysis and serves as a check on the formulation and numerical work.

The total heat transfer per meter of duct is calculated as

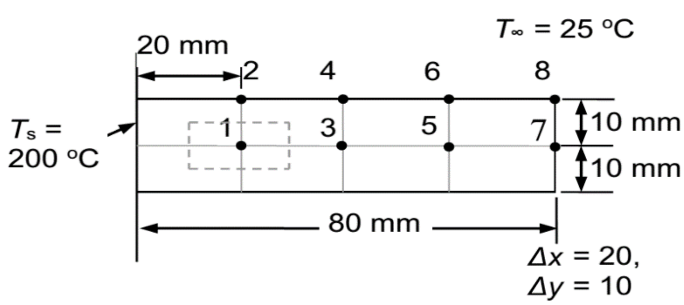
$$\varphi = 8(8,415)(1,21)$$
$$= 826,8\left[W/m^2\right]$$

<u>EXERCISE 3.2</u>

The figure below illustrates a rectangular fin section of a furnace wall. The fin is assumed to be infinitely wide in the z direction, allowing for a two-dimensional heat transfer analysis. The fin is exposed to a convection environment characterized by a heat transfer coefficient h= 400 W/(m^2. K) and an ambient temperature T_∞= 25°C

The thermal conductivity of the fin material is 4 W/ (m K).

Using a numerical (finite-difference) method, calculate the steady-state temperatures for all the interior and boundary nodes shown in the figure.



Solution

Due to the symmetry about the horizontal centerline, there are only 8 different nodal conditions.

Interior nodes (1, 3 and 5)

At node 1, the heat balance gives

$$\lambda A_y \frac{(T_s - T_1)}{\Delta x} + \lambda A_y \frac{(T_3 - T_1)}{\Delta x} + 2\lambda A_x \frac{(T_2 - T_1)}{\Delta y} = 0$$

Where $A_x = (20 \times 1).10^{-3}\left[m^2\right]$ for depth =1m

$$A_y = (10 \times 1).10^{-3}\left[m^2\right]$$

$$\Delta x = (20).10^{-3}[m]$$
$$\Delta y = (10).10^{-3}[m]$$

Substitution of various values gives

$$10\lambda \frac{(T_s - T_1)}{20} + 10\lambda \frac{(T_3 - T_1)}{20} + 2 \times 20 \times \lambda \frac{(T_2 - T_1)}{10} = 0$$

Or $\dfrac{(T_s - T_1)}{2} + \dfrac{(T_3 - T_1)}{2} + 4(T_2 - T_1) = 0$

Or $10T_1 - T_s - T_3 - 8T_2 = 0$

So the nodal equation is: $10T_1 - 8T_2 - T_3 - 200 = 0$

Similarly, the equation for the nodes 3 and 5 can be written as:

$$10T_3 - 8T_4 - T_5 - T_1 = 0$$
$$10T_5 - 8T_6 - T_7 - T_3 = 0$$

Nodes 2, 4, and 6:

These nodes are on the convective boundary. At node 2, the heat balance equation is

$$\lambda A_x \frac{(T_1 - T_2)}{\Delta y} + \lambda \frac{A_y}{2} \frac{(T_4 - T_2)}{\Delta x} + \lambda \frac{A_y}{2} \frac{(T_s - T_2)}{\Delta y} + h(\Delta x.1)(T_\infty - T_2) = 0$$

Putting values of $\Delta x, \Delta y, A_x, A_y$ and *h*, we obtain:

$$\left[4\times 20\left(10^{-3}\right)\right]\frac{(T_1 - T_2)}{10\left(10^{-3}\right)} + \left[4\times 10\left(10^{-3}\right)\right]\frac{(T_4 - T_2)}{2\times 20\left(10^{-3}\right)} + \left[4\times 10\left(10^{-3}\right)\right]\frac{(T_s - T_2)}{2\times 20\left(10^{-3}\right)}$$
$$+\left[400\times\left(20\left(10^{-3}\right)\right)\right](T_\infty - T_2) = 0$$

Or $8(T_1 - T_2) + (T_4 - T_2) + (T_s - T_2) + 8(T_\infty - T_2) = 0$

$$18T_2 - 8T_1 - T_4 - T_s - 8T_\infty = 0$$

Substituting values of T_s and T_∞, the nodal equation is :

$$18T_2 - 8T_1 - T_4 - 400 = 0$$

Similarly, at node 4, we have:

$$18T_4 - 8T_3 - T_6 - T_2 - 200 = 0$$

And the node 6:

$$18T_6 - 8T_5 - T_8 - T_4 - 200 = 0$$

Node 8 (the corner node):

$$\lambda \frac{A_y}{2}\frac{\left(T_6-T_8\right)}{\Delta x}+\lambda \frac{A_x}{2}\frac{\left(T_7-T_8\right)}{\Delta y}+h\frac{\left(\Delta x+\Delta y\right)}{2}.1.\left(T_\infty-T_8\right)=0$$

Then

$$4\times 10\times 10^{-3}\frac{\left(T_6-T_8\right)}{2\times 20\times 10^{-3}}+4\times 20\times 10^{-3}\frac{\left(T_7-T_8\right)}{2\times 10\times 10^{-3}}+400\frac{\left(20+10\right)10^{-3}}{2}\left(T_\infty-T_8\right)=0$$

Or $\left(T_6-T_8\right)+4\left(T_7-T_8\right)+6\left(T_\infty-T_8\right)=0$

Hence $11\ T_8-4\ T_7-T_6-150=0$

<u>Node 7 at the fin end:</u>

$$\lambda A_y\frac{\left(T_5-T_7\right)}{\Delta x}+2\lambda \frac{A_x}{2}\frac{\left(T_8-T_7\right)}{\Delta y}+h\left(\Delta y.1\right)\left(T_\infty-T_7\right)=0$$

Substitution of various values gives:

$7\ T_7-4\ T_8-T_5-50=0$

The complete set of nodal equations is

Node 1: $10T_1-8T_2-T_3-200=0$
Node 2: $18T_2-8T_1-T_4-400=0$
Node 3: $10T_3-8T_4-T_5-T_1=0$
Node 4: $18T_4-8T_3-T_6-T_2-200=0$
Node 5: $10T_5-8T_6-T_7-T_3=0$
Node 6: $18T_6-8T_5-T_8-T_4-200=0$
Node 7: $7\ T_7-4\ T_8-T_5-50=0$
Node 8: $11\ T_8-4\ T_7-T_6-150=0$

The above set of equations can be solved by Gaussian elimination of relaxation method.

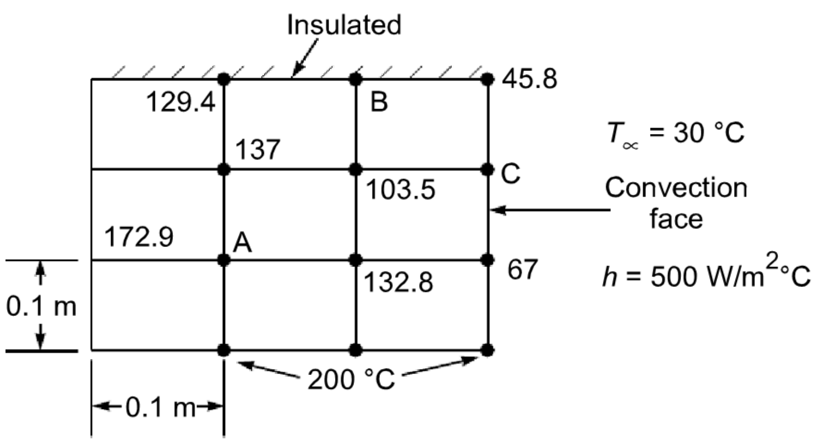
The relaxation method yields

$$T_1 = 65.91\left[{}^\circ C\right], T_2 = T_9 = 53.2\left[{}^\circ C\right], T_3 = 33.69\left[{}^\circ C\right], T_4 = T_{10} = 30.5\left[{}^\circ C\right], T_5 = 26.77\left[{}^\circ C\right],$$
$$T_6 = T_{11} = 26.1\left[{}^\circ C\right], T_7 = 25.36\left[{}^\circ C\right], T_8 = T_{12} = 25.22\left[{}^\circ C\right],$$

EXERCICE 3.3

Analyze the steady-state nodal temperatures for the solid section shown in the figure, which includes fixed temperature, insulated, and convection boundaries. We Giveλ=1.5 (W/m.K) and h=500 (W/m^2.K).

calculate the temperatures at T_A, T_B and T_C and the total heat transfer rate from the surface subjected to convection.



For node 'A', we have

$$T_A = \frac{T_1 + T_4 + T_5 + T_7}{4}$$
$$= \frac{200 + 200 + 132.8 + 137}{4} = 160.7°C$$

For node 'B', we have

$$T_B = \frac{T_9 + 2T_8 + T_{10}}{4}$$
$$= \frac{129.4 + 2(103.5) + 45.8}{4} = 95.6°C$$

For node 'C', we have

$$T_C = \frac{0,5(T_6 + 2T_8 + T_{10}) + \frac{h.\Delta x}{\lambda} T_\infty}{\frac{h.\Delta x}{\lambda} T_\infty}$$
$$= \frac{0,5(67 + 2(103) + 45,8) + \frac{500.(0,1)}{1,5} 30}{\frac{500.(0,1)}{1,5} 30 + 2} = 37,4°C$$

Now the heat convected out by the exposed surface is

$$\phi = hA.\Delta T$$
$$= h.(\Delta y.\Delta z)\sum(T - T\infty),$$
$$= 500(1\text{x}0,1)\left[(45,8-30)+(37,4-30)+(67-30)+2(200-30)\right]$$
$$= 7258[W]$$

CHAPTER 4: VARIABLE REGIME HEAT CONDUCTION

4.1 INTRODUCTION

Until now, the heat conduction models discussed have focused exclusively on temperature variation as a function of position. However, in many engineering applications, the spatial temperature variation within a medium is negligible, allowing temperature to be treated as a function of time only. These simplified formulations, known as the lumped system or lumped capacitance method, greatly simplify transient heat conduction analysis but their field of application is very limited. We will illustrate this approach and examine its validity based on Biot dimensionless number.

4.1 SYSTEMS WITH NEGLIGIBLE INTERNAL RESISTANCE .

Lumped and partially lumped formulation

Let's analyze a small solid composed of a material with high thermal conductivity, Initially, this solid has a uniform temperature T_0. It is then suddenly immersed into a hot fluid bath that is well-stirred and maintained at a uniform temperature T_∞. This solid has a volume V, a specific heat C_p. and a

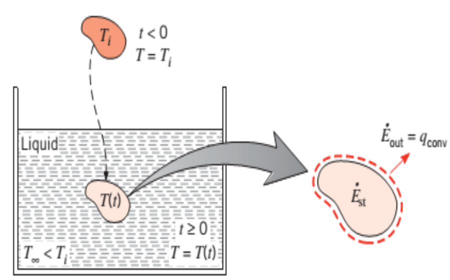


Figure 4.1: Heating of a metal

a density ρ a surface area A through which heat can be exchanged (Figure 4.1)

The heat transfer between the object's surface and the surrounding fluid is characterized by the convection heat transfer coefficient h. Because the solid is small and highly conductive, we can assume that the temperature within it remains essentially uniform throughout at any given time, meaning the temperature T is only a function of time T(t). By considering the entire solid object as our control volume, the principle of energy conservation can be expressed as :

(Rate of heat flow from the solid through its boundaries) = (Rate of change of the internal energy of the solid) (4.1)

When heat transfer to or from the control volume occurs only via convection, the energy equation is expressed by:

$$\rho V C_p \frac{dT}{dt} = -hA\left[T(t) - T_\infty\right] \tag{4.2}$$

which is rearranged to yield

$$\frac{dT}{dt} + \frac{hA}{\rho V C_p}\left[T(t) - T_\infty\right] = 0 \quad \text{for} \quad t > 0 \tag{4.3}$$

Initial condition: $T(t=0) = T_0$ (4.4)

Equation (4.3) is a nonhomogeneous ordinary differential equation that can be solved by finding the sum of its homogeneous and particular solutions. Nevertheless, to simplify the equation, it is useful to define the temperature θ (t) as follows:

$$\theta(t) = T(t) - T_\infty \tag{4.5}$$

With this substitution, the lumped formulation becomes:

$$\begin{aligned} &\frac{d\theta(t)}{dt} + \frac{1}{b}.\theta(t) = 0 \quad \text{for t >0} \\ &\theta(t=0) = T_0 - T_\infty = \theta_0 \end{aligned} \tag{4.6}$$

Where $b = \dfrac{\rho V C_p}{hA}$ (4.7)

The solution of equations (4.6) is :

$$\theta(t) = \theta_0 e^{-\frac{t}{b}} \tag{4.8}$$

This equation provides a very simple way to determine the temperature of the solid at any given time. It's important to note that the parameter b in this equation has units of seconds (s), which is known as the thermal time constant of the system.

This term represents the product of the heat capacity multiplied by the convective thermal resistance. Consequently, if either the heat capacity or the convective resistance (or both) is smaller, the value of b will be larger, and equation (4.8) shows that the solid temperature θ(t) will change more rapidly.

Temperature evolution of this system is presented in figure 4.2.

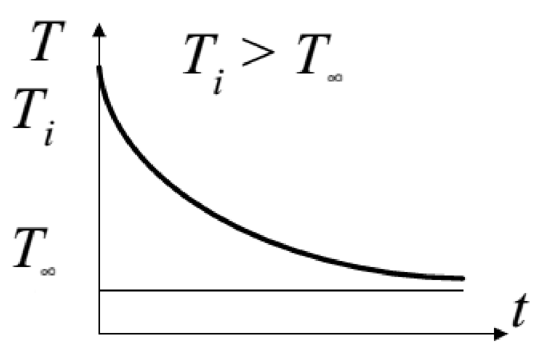


Figure 4.2: Evolution of temperature

To determine the applicability criterion for transient heat conduction analysis, we will introduce the definition of the Biot number which is given by the following form :

$$Bi = \frac{hL_c}{\lambda_s} = \frac{L_c / \lambda_s A}{1/hA} = \frac{\text{internal conductive resistance}}{\text{external convective resistance}} \tag{4.9}$$

Where λ_s is the thermal conductivity of the solid. L_c is the characteristic length of the solid ,it is generally defined as $L_c = V/A$.

The lumped system analysis is a simplification used in transient heat transfer problems. It relies on the assumption that the temperature within a solid body remains sufficiently uniform during the heat transfer process.

This assumption is valid when the thermal resistance within the body is negligible compared to the thermal resistance at its surface. This condition is checked using Biot number (Bi).

Referring to the Biot number definition, a small Biot number (significantly less than one, like an order of magnitude smaller) indicates this condition. Consequently, the lumped system analysis is valid only for small values of the Biot numbers, specifically [21]:

$$Bi = \frac{hL_c}{\lambda_s} < 0.1 \quad \text{(Lumped analysis criterion)} \tag{4.10}$$

Equation (4.10) presents a valuable criterion for applying lumped analysis. However, for transient problems, however, the reduction of spatial temperature gradients is a gradual process that happens as the Biot number decreases. Therefore, we shouldn't view a Biot number of 0.1 as a strict limit for the existence or absence of these gradients. Instead, it's a practical criterion for satisfying the condition $Bi \ll 1$. This is supported by precise analytical solutions for transient heat transfer in walls, cylinders, and spheres with convective cooling, which show that internal temperature differences stay below roughly 5% when the Biot number is less than 0.1.

Therefore, in numerous engineering applications, lumped system analysis is generally acceptable when the Biot number is approximately less than 0.1. To a better understand the transient temperature distribution, figure 4.3 illustrates the temperature profiles for a symmetric plane wall. This wall, initially at a uniform temperature T_0 , undergoes convective cooling on both surfaces by a fluid at a temperature of T_∞ . This figure examines three different Biot number: $Bi \ll 1$, $Bi \approx 1$, and $Bi \gg 1$. When the Biot number is significantly less than 1 ($Bi \ll 1$), as shown in Figure 4.3.a, the plane wall undergoes a uniform temperature decrease from the initial value T_0 towards the steady-state fluid temperature T_∞ .

This behavior is typical of situations where the resistance to heat transfer at the surface (convective resistance) is significantly greater than the resistance to heat transfer within the wall (conductive resistance). This dynamic leads to a uniform temperature throughout the wall at any given moment, meaning the temperature is a function of time alone, $T=T(t)$.

Figure 4.3.b illustrates the intermediate case where $Bi \approx 1$. Here, the conductive resistance within the wall and the convective resistance at the surface are comparable, leading to the variations of temperature across the wall's thickness and a temperature difference between the wall and the fluid.

Finally, figure 4.3.c illustrates the case where Bi is much greater than 1 ($Bi \gg 1$). In the limiting scenario, this approaches an infinitely large convection coefficient ($h \rightarrow \infty$), which effectively imposes a fixed surface temperature. In this regime, spatial temperature gradients are the dominant factor. Specifically, the gradient is exceptionally steep near the surfaces at $t=0$.

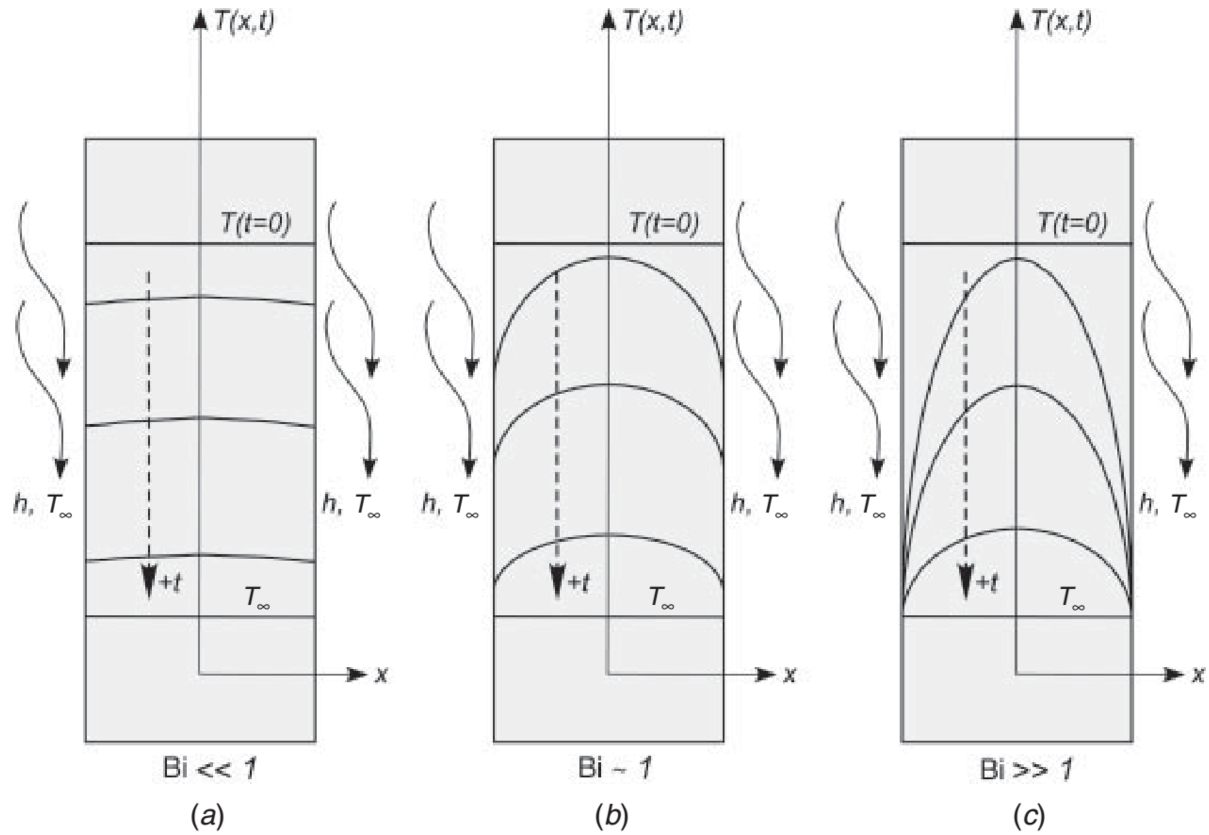
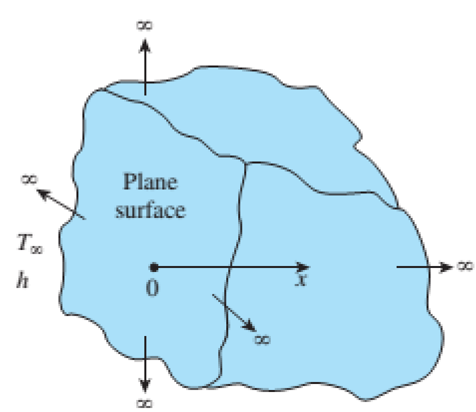


Figure 4.3: Temperature distribution T(x,t)for a symmetric plane wall cooled by convection heat transfer for various Biot numbers .

4.2 UNSTEADY-STATE CONDUCTION IN INFINITE MEDIUM.

<u>Tabulated solutions</u>

In this section, we will introduce solutions to unsteady-state (or transient) problems, where temperature changes with time.



We Consider a semi-infinite solid that occupies the space $x \geq 0$, the entire solid is at a uniform temperature, T_0. At time t=0, the surface at x=0 is instantaneously and permanently changed to a new constant temperature T_s.

We assume that the temperature is uniform across the y and z directions at all times, meaning that heat transfer occurs only along the x-axis. This one-dimensional heat flow scenario can be mathematically represented by considering the solid to extend infinitely in both the positive and negative y and z directions.

When the surface temperature of a solid, T_s, is higher than its initial uniform temperature, T_0, heat will diffuse into the material. This causes the temperature at any point within the solid to increase over time. Consequently, the temperature T, is a function of both position (x) and time (t), making it an unsteady-state (or transient) heat transfer problem. The primary objective is to find this temperature distribution T(x, t).

When internal heat generation is negligible and the thermal conductivity is constant, the heat conduction equation simplifies to the form :

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{4.11}$$

The boundary conditions are:

$$\begin{aligned} &-At\ t = 0,\ T = T_0 \text{ for all } x \geq 0 \\ &-At\ x = 0,\ T = T_s \text{ for all t} \succ 0 \\ &-As\ x \to \infty,\ T \to T_0 \text{ for all t} \geq 0 \end{aligned} \tag{4.12}$$

This final condition is physically intuitive: heat requires an infinite amount of time to propagate through an infinitely large distance within the solid.

We can solve equation (4.11) with these boundary conditions using the method of combination of variables, yielding the following result:

$$\frac{T(x,t)-T_s}{T_0-T_s}=erf\left(\frac{x}{2\sqrt{\alpha t}}\right) \tag{4.13}$$

The error function, erf, is defined by:

$$erf\left(\frac{x}{2\sqrt{\alpha t}}\right)=\frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2\sqrt{\alpha t}}} e^{-z^2}\,dz \tag{4.14}$$

This function is used in various engineering and applied science applications, it can be evaluated through numerical integration. Its computed values are provided in Table 1.4.

*

Table 4.1: The error function.

X	erf x	X	erf x	X	erf x
0.00	0.00000	0.76	0.71754	1.52	0.96841
0.02	0.02256	0.78	0.73001	1.54	0.97059
0.04	0.04511	0.80	0.74210	1.56	0.97263
0.06	0.06762	0.82	0.75381	1.58	0.97455
0.08	0.09008	0.84	0.76514	1.60	0.97635
0.10	0.11246	0.86	0.77610	1.62	0.97804
0.12	0.13476	0.88	0.78669	1.64	0.97962
0.14	0.15695	0.90	0.79691	1.66	0.98110
0.16	0.17907	0.92	0.80677	1.68	0.98249
0.18	0.20094	0.94	0.81627	1.70	0.98379
0.20	0.22270	0.96	0.82542	1.72	0.98500
0.22	0.24430	0.98	0.83423	1.74	0.98613
0.24	0.26570	1.00	0.84270	1.76	0.98719
0.26	0.28690	1.02	0.85084	1.78	0.98817
0.28	0.30788	1.04	0.85865	1.80	0.98909
0.30	0.32863	1.06	0.86614	1.82	0.98994
0.32	0.34913	1.08	0.87333	1.84	0.99070
0.34	0.36936	1.10	0.88020	1.86	0.99147
036	0.38933	1.12	0.88079	1.88	0.99216
0.38	0.40901	1.14	0.89308	1.90	0.99279
0.40	0.42839	1.16	0.89910	1.92	0.99338
0.42	0.44749	1.18	0.90484	1.94	0.99392
0.44	0.46622	1.20	0.91031	1.96	0.99443
0.46	0.48466	1.22	0.91553	1.98	0.99489
0.48	0.50275	1.24	0.92050	2.00	0.995322
0.50	0.52050	1.26	0.92524	2.10	0.997020
0.52	0.5379	1.28	0.92973	2.20	0.998137
0.54	0.55494	1.30	0.93401	2.30	0.998857
0.56	0.57162	1.32	0.93806	2.40	0.999311
0.58	0.58792	1.34	0.94191	2.50	0.999593
0.60	0.60386	1.36	0.94556	2.60	0.999764
0.62	0.61941	1.38	0.94902	2.70	0.999866
0.64	0.63465	1.40	0.95228	2.80	0.999925
0.66	0.64938	1.42	0.95538	2.90	0.999959
0.68	0.66278	1.44	0.95830	3.00	0.999978
0.70	0.67780	1.46	0.96105	3.20	0.999994
0.72	0.69143	1.48	0.96365	3.40	0.999998
0.74	0.70468	1.50	0.96610	3.60	1.000000

The heat flux is given by:

$$\varphi_x = \frac{\lambda\left(T_s - T_0\right)}{\sqrt{\pi\alpha t}}\exp\left(-x^2 / 4\alpha t\right) \tag{4.15}$$

The total amount of heat transferred per unit area across the surface at x = 0 over time t is given by the following equation:

$$\frac{Q}{A} = 2\lambda\left(T_s - T_0\right)\sqrt{\frac{t}{\pi\alpha}} \tag{4.16}$$

The concept of a semi-infinite solid is a valuable simplification used to solve complex heat transfer problems in various practical situations. For instance, the Earth can be effectively modeled as a semi-infinite solid.

A solid of any actual thickness can be accurately modeled as a semi-infinite solid provided that the duration of heat transfer is brief enough that the thermal disturbance, or penetration depth, affects only a small portion of the object near its surface. This simplification is highly valuable because it allows for the analysis of transient heat transfer problems without needing to consider the object's full dimensions. Despite its seemingly abstract nature, the semi-infinite solid concept is a powerful tool with numerous applications in engineering and science.

The acceptability of this approximation is usually determined by the following inequality:

$$\frac{\alpha t}{L^2} < 0.1 \tag{4.17}$$

Where L is the thickness of the solid. The dimensionless group $\alpha t / L^2$ is called the Fourier number and is designated as F_o or τ .

4.3 VARIABLE SEPARATION METHOD IN FINITE MEDUIM

We consider a rod of length 2L that is initially at a uniform temperature T_0 . At time t=0, the temperatures at both ends of the rod, at x=0 and x=2L, are suddenly changed and held constant at a new temperature T_1, as shown in figure 4.4 .

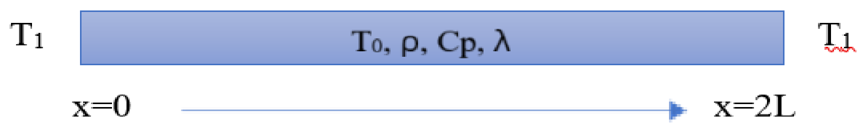


Figure 4.4: rod study .

The governing differential equation of this problem is :

$$\lambda \frac{\partial^2 T}{\partial x^2} = \rho C_p \frac{\partial T}{\partial t} \qquad (4.18)$$

The form of this equation shows that it is a linear differential equation of second order in space x and first order in time t.

We reformulate this equation by putting $\alpha = \frac{\lambda}{\rho C_p}$ and $\theta = T - T_1$, we obtain :

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha} \frac{\partial \theta}{\partial t} \qquad (4.19)$$

Using this new form, we examine the boundary conditions for $\theta(x,t)$:

For $0 \le x \le 2L$ and $t = 0$ $\qquad \theta(x,0) = T_0 - T_1 \qquad$ (4.20.a)

For $x = 0$ and $t > 0$ $\qquad \theta(0,t) = 0 \qquad$ (4.20.b)

For $x = 2L$ and $t > 0$ $\theta(2L, t) = 0$ (4.20.c)

To solve this differential equation using the method of variables separation, we assume a solution of the form :

$$\theta(x,t) = X(x).\tau(t) \qquad (4.21)$$

Where:

X(x) is a function of x only, and, $\tau(t)$ is a function of t only.

Calculating the derivatives of $\theta(x,t)$ gives us:

$$\begin{aligned} \frac{\partial^2 \theta(x,t)}{\partial x^2} &= \frac{\partial^2 X(x)}{\partial x} \tau(t) \\ \frac{\partial \theta(x,t)}{\partial t} &= \frac{\partial \tau}{\partial t} X(x) \end{aligned} \qquad (4.22)$$

Thus, the transformed differential equation becomes:

$$\frac{\partial^2 X(x)}{\partial x^2} \tau(t) = \frac{X(x)}{\alpha} \frac{\partial \tau(t)}{\partial t} \qquad (4.23)$$

We can rearrange the previous equation as follows :

$$\frac{1}{X} \frac{\partial^2 X}{\partial x^2} = \frac{1}{\alpha.\tau} \frac{\partial \tau}{\partial t} \qquad (4.24)$$

To facilitate solution, we can make both equations equal to a constant K which is called the separation constant :

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}=\frac{1}{\alpha.\tau}\frac{\partial \tau}{\partial t}=-K^2 \tag{4.25}$$

We can now write two independent linear differential equations:

$$\begin{aligned}&\frac{\partial^2 X}{\partial x^2}+K^2 X\left(x\right)=0\\&\frac{\partial \tau}{\partial t}+K^2\alpha.\tau\left(t\right)=0\end{aligned} \tag{4.26}$$

The first equation:

$$\frac{\partial^2 X}{\partial x^2}+K^2 X\left(x\right)=0 \tag{4.27}$$

has a solution of the form:

$$X\left(x\right)=C_1 cos(Kx)+C_2\sin(Kx) \tag{4.28}$$

The second equation:

$$\frac{\partial \tau}{\partial t}+K^2\alpha.\tau\left(t\right)=0 \tag{4.29}$$

has a solution of the form

$$\tau\left(t\right)=C_3\exp(-K^2\alpha.t) \tag{4.30}$$

The expression of $\theta\left(x,t\right)$ becomes:

$$\theta\left(x,t\right)=X(x).\tau(t)=\left[C_1 cos(Kx)+C_2\sin(Kx)\right].C_3\exp(-K^2\alpha.t) \tag{4.31}$$

We can simplify this expression by taking:

$$A_1 = C_1.C_3$$
$$A_2 = C_2.C_3 \tag{4.32}$$

The expression 4.31 becomes:

$$\theta(x,t) = \left[A_1 cos(Kx) + A_2 \sin(Kx)\right].\exp(-K^2\alpha.t) \tag{4.33}$$

Using the boundary conditions established earlier, we can now determine the constants A_1 and A_2.

When we apply the second boundary condition (equation 4.20.b), we obtain:

$$\theta(0,t) = \left[A_1 cos(0) + A_2 \sin(0)\right].\exp(-K^2\alpha.t) = 0 \tag{4.34}$$

We can deduce that A_1 must be equal to zero.

$$\theta(x,t) = A_2 \sin(Kx)\exp(-K^2\alpha.t) \tag{4.35}$$

It is clear that A_2 cannot be equal to zero; otherwise, $\theta(x,t) = 0$ throughout the domain.

Application of the third condition to the limits (equation 4.20.c) yields:

$$\theta(2L,t) = A_2 \sin(2KL)\exp(-\lambda^2 K.t) = 0 \tag{4.36}$$

Since $A_2 \neq 0$, therefore:

$$\sin(2KL) = 0 \tag{4.37}$$

This condition is checked if and only if:

$$K = \frac{\pi n}{2L} \tag{4.38}$$

So, the solution may be expressed in the form of a series given by :

$$\theta(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{2L}\right) \exp\left[-\left(\frac{n\pi}{2L}\right)^2 \alpha.t\right] \tag{4.39}$$

$$A_n = \frac{1}{L}\int_0^{2L} (T_0 - T_1) \sin\left(\frac{n\pi x}{2L}\right) dx \qquad ; n = 1, 3, 5.... \tag{4.40}$$

A_n may be determined by introducing the initial conditions (for n = 1, 3, 5, ...)

$$A_n = \frac{1}{L}\int_0^{2L} (T_0 - T_1) \sin\left(\frac{n\pi x}{2L}\right) dx = \frac{4}{n\pi}(T_0 - T_1) \tag{4.41}$$

The overall solution becomes (for n = 1, 3, 5, ...):

$$\frac{T - T_1}{T_0 - T_1} = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi x}{2L}\right) \exp\left[-\left(\frac{n\pi}{2L}\right)^2 \alpha.t\right] \tag{4.42}$$

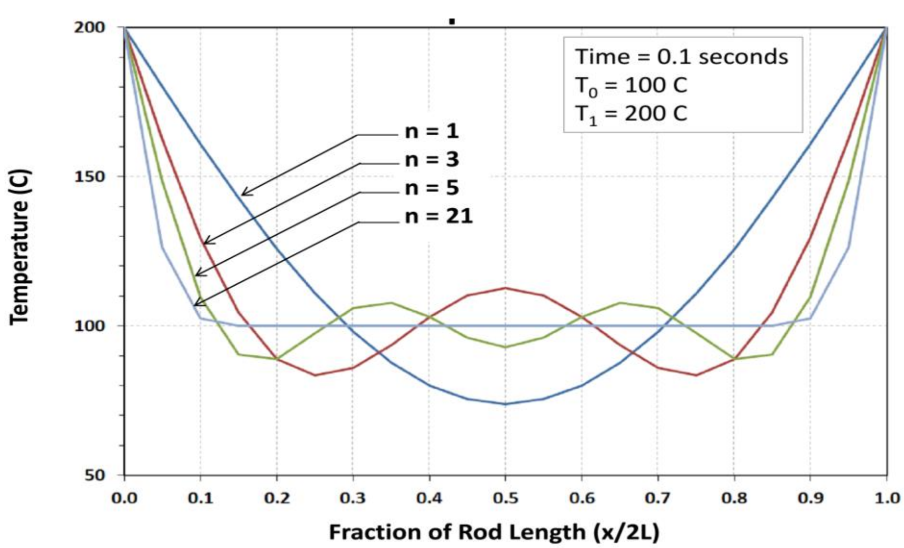


Figure 4.5: Solving the transient heat equation for t=0.1s

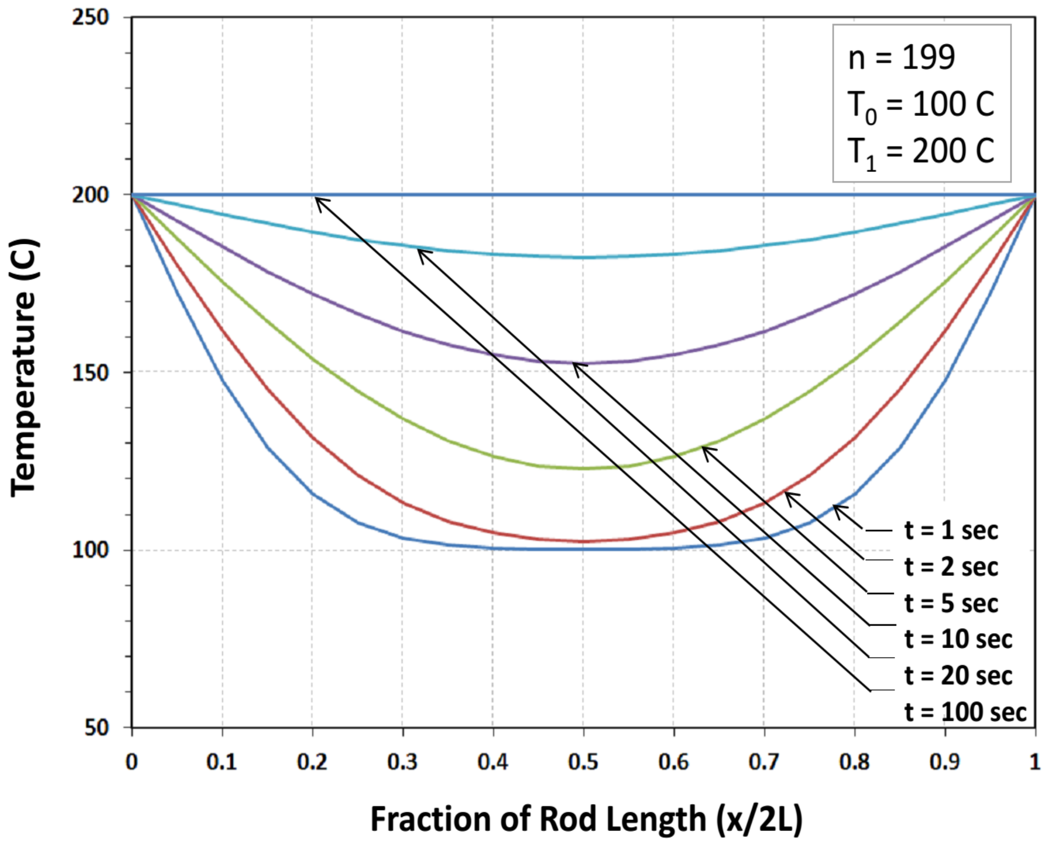


Figure 4.6: Solving the transient heat equation for n=199

4.4 TRANSIENT HEAT CONDUCTION IN LARGE PLANE WALLS, LONG CYLINDERS, AND SPHERES WITH SPATIAL EFFECTS

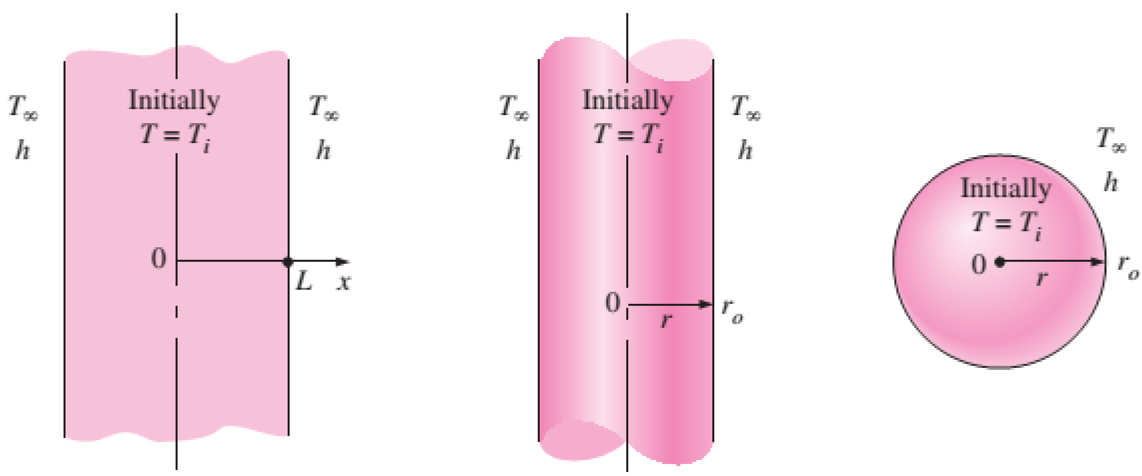
Abacus method

This section examines the temporal and spatial temperature variations in one-dimensional systems like a large plane wall, a long cylinder, and a sphere. We consider these geometries with a plane wall of thickness 2L and a cylinder and sphere, both with a radius of r_0

Initially, each of these bodies is at a uniform temperature T_i for (t=0) as illustrated in Figure 4.7, they are suddenly introduced into a vast surrounding medium maintained at a constant temperature T_∞. Convection heat transfer occurs between these bodies and their surroundings, characterized by a uniform and constant heat transfer coefficient h. Notably, each case demonstrates inherent geometric and thermal symmetry; the plane wall, for example, is symmetric about its central plane (x=0). The cylinder is symmetrical along its central axis (r=0), and the sphere is symmetrical around its central point (r=0).

Figure 4.8 illustrates the evolution of the temperature distribution within a plane wall over time. Initially, at t=0, the entire wall is at a uniform temperature T_i. It is then subjected to a surrounding medium at a constant temperature T_∞. (the temperature decrease implies that $T_\infty < T_i$).

However, as heat is transferred from the wall to the surroundings, the temperature at and near the surfaces begins to decrease.



(a) A large plane wall (b) A long cylinder (c) A sphere

Figure 4.7: schematic of the simple geometric configurations that result in one-dimensional heat transfer

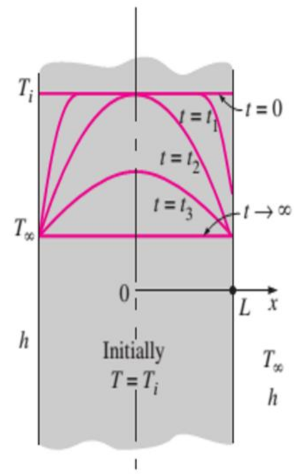


Figure 4.8: Transient temperature .

Consequently, a temperature gradient is developed across the wall's thickness, causing heat to conduct from the warmer inner regions toward the cooler outer surfaces. It's notable that, at the initial moments, the temperature at the centre of the wall remains at T_i because it takes time for the thermal effects from the surface to reach the core.

The initial uniform temperature at the centre of the wall is maintained until a later time, t2. Throughout this process, the temperature distribution remains symmetric about the central plane. As heat transfer continues, the temperature profile becomes progressively flatter, eventually approaching a uniform temperature T_∞. At this point, the wall achieves thermal equilibrium with its surroundings, and heat transfer ceases due to the absence of a temperature gradient. Analogous behaviors are also observed for the long cylinder and the sphere.

The analysis of the one-dimensional, unsteady temperature distribution, T(x,t), in a wall requires the formulation and solution of a partial differential equation (PDE), a task that typically involves advanced mathematical techniques.

While a rigorous mathematical solution exists, it typically involves an infinite series that can be difficult and time-consuming to calculate. For this reason, it is necessary to present the solution in a more accessible format, such as tables or graphs.

However, the solution to this transient heat transfer problem depends on numerous physical parameters and variables (x, L, t, λ, ρ, Cp, h, T_i, T_∞), making a practical graphical representation challenging.

To overcome this difficulty, the parameters are combined into dimensionless numbers. Among this numbers, we can mention: [1]

Dimensionless temperature: $\theta\left(x,t\right)=\dfrac{T\left(x,t\right)-T_\infty}{T_i-T_\infty}$

Dimensionless distance from the center: $X=\dfrac{x}{L}$

Dimensionless heat transfer coefficient: $Bi=\dfrac{hL_c}{\lambda}$ (Biot number)

Dimensionless time $\tau=\dfrac{\alpha t}{{L_c}^2}$ (Fourier number)

Nondimensionalizing the variables is a powerful method for simplifying the analysis of transient heat transfer. The solution can be expressed as a function of only three dimensionless parameters: X, Bi, and τ. This significant reduction in variables makes it feasible to present the solution graphically. It should be noted that the dimensionless quantities defined for a plane wall can be adapted for cylinders and spheres by substituting the spatial variable x with the radial variable r and the half-thickness L with the outer radius r_o.

It's important to note that the definition of characteristic length for the Biot number differs between these two types of analysis. It is the half-thickness for a plane wall L, and for a cylinder or sphere, it is the radius r_0 , unlike the V/A ratio used as the characteristic length in lumped capacitance methods. This distinction is crucial for correctly applying the analytical solutions and charts for these geometries.

Although the exact solution to one-dimensional transient heat conduction problems involves a mathematically complex infinite series, a much more practical approach is often used. A key feature of these series is their rapid convergence over time. For this reason, when the Fourier number τ is 0.2 or greater, using only the first term of the series provides a highly accurate approximation, with an error of less than 2 percent.

For problems where the Fourier number τ is 0.2 or greater, the infinite series solution converges rapidly, making it very convenient and accurate to use a one-term approximation. The general solution is simply the first term of the series, given by the following equations for each geometry :

Plane wall $$\theta(x,t)_{\text{wall}} = \frac{T(x,t) - T_\infty}{T_i - T_\infty} = A_1\ e^{-\lambda_1^2 \tau} \cos(\lambda_1 x / L), \tau > 0.2 \quad (4.43)$$

Cylinder $$\theta(r,t)_{\text{cyl}} = \frac{T(r,t) - T_\infty}{T_i - T_\infty} = A_1\ e^{-\lambda_1^2 \tau} J_0(\lambda_1 r / r_0), \tau > 0.2 \quad (4.44)$$

Sphere $$\theta(r,t)_{\text{sph}} = \frac{T(r,t) - T_\infty}{T_i - T_\infty} = A_1\ e^{-\lambda_1^2 \tau} \frac{\sin(\lambda_1 r / r_0)}{(\lambda_1 r / r_0)}, \tau > 0.2 \quad (4.45)$$

In this one-term approximation, the constants A1 and λ_1 are dependent only on the Biot number (Bi). Their corresponding values for plane walls, cylinders, and spheres are typically provided in tables, such as Table 4.2, for various Biot numbers.

Table 4.2. Coefficient used in the one term approximate solution of transient one-dimensional heat conduction

	Plane Wall		*Cylinder*		*Sphere*	
Bi	λ_1(rad)	A_1	λ_1(rad)	A_1	λ_1(rad)	A_1
0.01	0.0998	1.0017	0.1412	1.0025	0.1730	1.0030
0.02	0.1410	1.0033	0.1995	1.0050	0.2445	1.0060
0.04	0.1987	1.0066	0.2814	1.0099	0.3450	1.0120
0.06	0.2425	1.0098	0.3438	1.0148	0.4217	1.0179
0.08	0.2791	1.0130	0.3960	1.0197	0.4860	1.0239
0.1	0.3111	1.0161	0.4417	1.0246	0.5423	1.0298
0.2	0.4328	1.0311	0.6170	1.0483	0.7593	1.0592
0.3	0.5218	1.0450	0.7465	1.0712	0.9208	1.0880
0.4	0.5932	1.0580	0.8516	1.0931	1.0528	1.1164
0.5	0.6533	1.0701	0.9408	1.1143	1.1656	1.1441
0.6	0.7051	1.0814	1.0184	1.1345	1.2644	1.1713
0.7	0.7506	1.0918	1.0873	1.1539	1.3525	1.1978
0.8	0.7910	1.1016	1.1490	1.1724	1.4320	1.2236
0.9	0.8274	1.1107	1.2048	1.1902	1.5044	1.2488
1.0	0.8603	1.1191	1.2558	1.2071	1.5708	1.2732
2.0	1.0769	1.1785	1.5995	1.3384	2.0288	1.4793
3.0	1.1925	1.2102	1.7887	1.4191	2.2889	1.6227
4.0	1.2646	1.2287	1.9081	1.4698	2.4556	1.7202
5.0	1.3138	1.2403	1.9898	1.5029	2.5704	1.7870
6.0	1.3496	1.2479	2.0490	1.5253	2.6537	1.8338
7.0	1.3766	1.2532	2.0937	1.5411	2.7165	1.8673
8.0	1.3978	1.2570	2.1286	1.5526	2.7654	1.8920
9.0	1.4149	1.2598	2.1566	1.5611	2.8044	1.9106
10.0	1.4289	1.2620	2.1795	1.5677	2.8363	1.9249
20.0	1.4961	1.2699	2.2880	1.5919	2.9857	1.9781
30.0	1.5202	1.2717	2.3261	1.5973	3.0372	1.9898
40.0	1.5325	1.2723	2.3455	1.5993	3.0632	1.9942
50.0	1.5400	1.2727	2.3572	1.6002	3.0788	1.9962
100.0	1.5552	1.2731	2.3809	1.6015	3.1102	1.9990
∞	1.5708	1.2732	2.4048	1.6021	3.1416	2.0000

The term J_0 represents the zeroth-order Bessel function of the first kind, and its values can be evaluated from the Table 4.3.

Table 4.3: The zeroth and first order Bessel Function of the first kind

η	$J_0(\eta)$	$J_1(\eta)$
0.0	1.0000	0.0000
0.1	0.9975	0.0499
0.2	0.9900	0.0995
0.3	0.9776	0.1483
0.4	0.9604	0.1960
0.5	0.9385	0.2423
0.6	0.9120	0.2867
0.7	0.8812	0.3290
0.8	0.8463	0.3688
0.9	0.8075	0.4059
1.0	0.7652	0.4400
1.1	0.7196	0.4709
1.2	0.6711	0.4983
1.3	0.6201	0.5220
1.4	0.5669	0.5419
1.5	0.5118	0.5579
1.6	0.4554	0.5699
1.7	0.3980	0.5778
1.8	0.3400	0.5815
1.9	0.2818	0.5812
2.0	0.2239	0.5767
2.1	0.1666	0.5683
2.2	0.1104	0.5560
2.3	0.0555	0.5399
2.4	0.0025	0.5202
2.6	−0.0968	0.4708
2.8	−0.1850	0.4097
3.0	−0.2601	0.3391
3.2	−0.3202	0.2613

Given that cos (0)=1 and $J_0(0)=1$, and knowing that $\lim_{x \to 0} (\text{sinx/x}) = 1$, these expressions simplify to the following forms at the centre of a plane wall, cylinder, or sphere:

Center of plane wall (x=0) $\theta_{0,\text{wall}} = \dfrac{T_0 - T_\infty}{T_i - T_\infty} = A_1 \ e^{-\lambda_1^2 \tau}$ (4.46)

Center of cylinder (r=0) $\theta_{0,\text{cyl}} = \dfrac{T_0 - T_\infty}{T_i - T_\infty} = A_1 \ e^{-\lambda_1^2 \tau}$ (4.47)

Center of sphere (r=0) $\theta_{0,\text{sph}} = \dfrac{T_0 - T_\infty}{T_i - T_\infty} = A_1 \ e^{-\lambda_1^2 \tau}$ (4.48)

With a known Biot number, the formulas presented above allow us to calculate the temperature at any point in the material. Determining the constants A_1 and λ_1 typically involves interpolation.

For users who prefer graphical interpretation, the above relations are often presented in the form of transient temperature charts, illustrating the one-term approximation solutions. however, it should be noted that while these charts offer a convenient visual approach, they can be difficult to read with high precision and are susceptible to interpretation errors. Consequently, the mathematical relations themselves are generally considered a more reliable method than using the charts called Heisler charts.

The transient temperature charts for a large plane wall, long cylinder, and sphere illustrated in (figures from 4.9 to 4.17) are attributed to M. P. Heisler (1947) and are referred to as Heisler charts, who developed them. These charts were later supplemented by H. Gröber in 1961. For each geometry, there are typically three charts: the first is used to determine the temperature at the center

(T_0) at a given time t, and the second chart helps determine the temperature at any other locations within the body at that same time, expressed in terms of T_0. The third chart is used to determine the total amount of heat transferred up to time t. These graphs are accurate for Fourier numbers (ҭ) of 0.2 or greater.

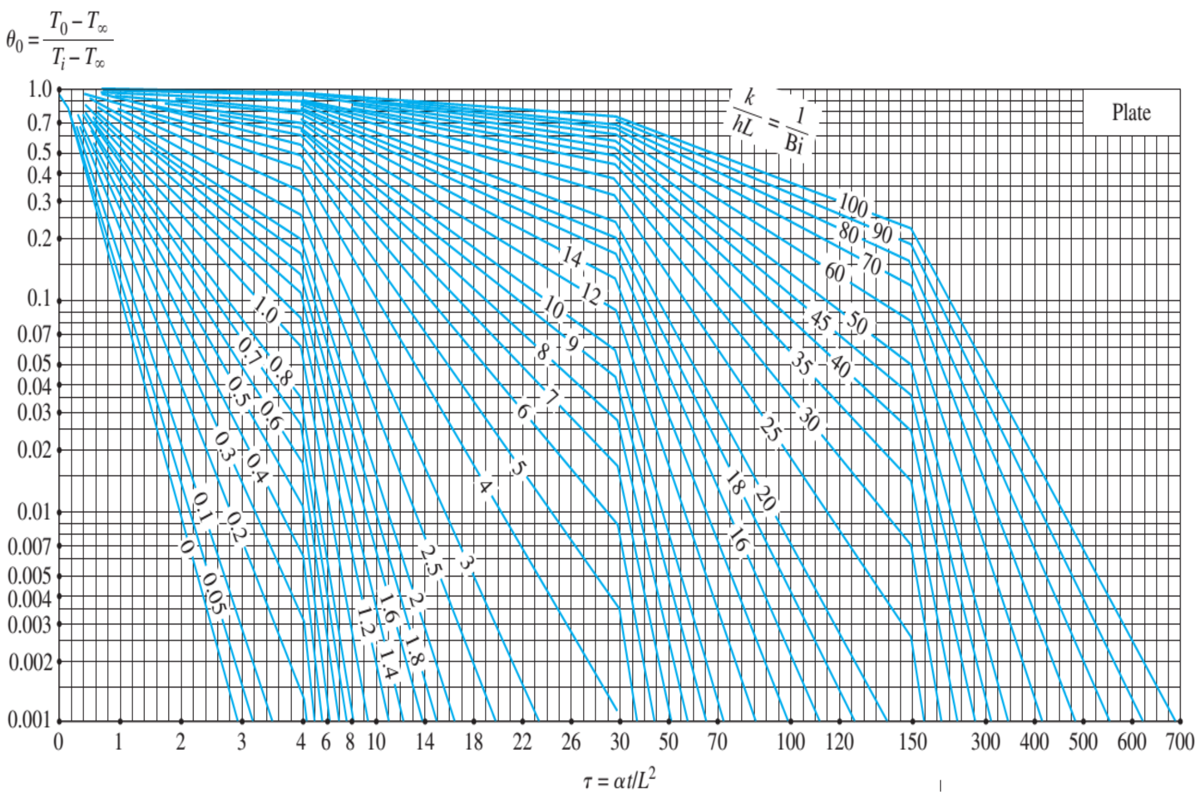


Figure 4.9: Midplane temperature in a plane wall .

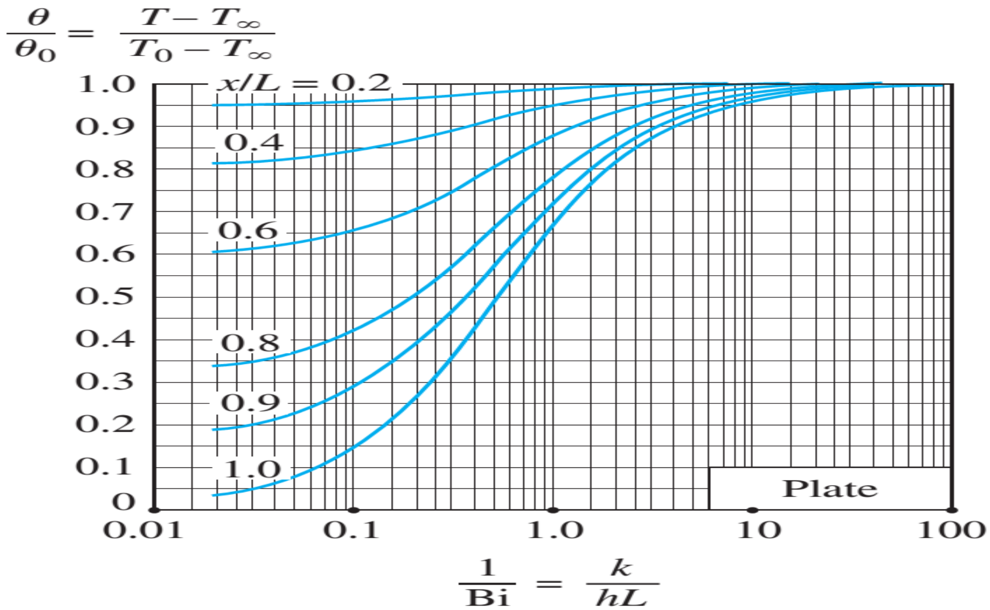


Figure 4.10: Centerline temperature in a plane wall.

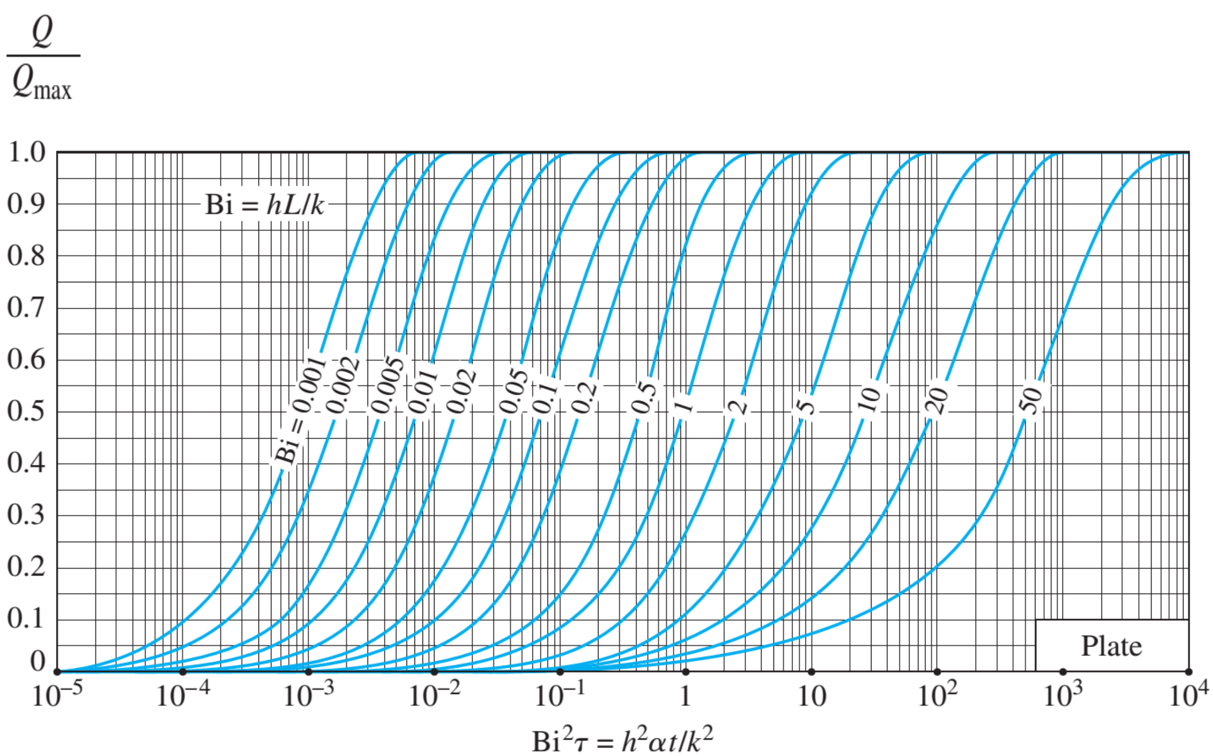


Figure 4.11: Dimensionless heat loss Q/Q_0 of a plane wall.

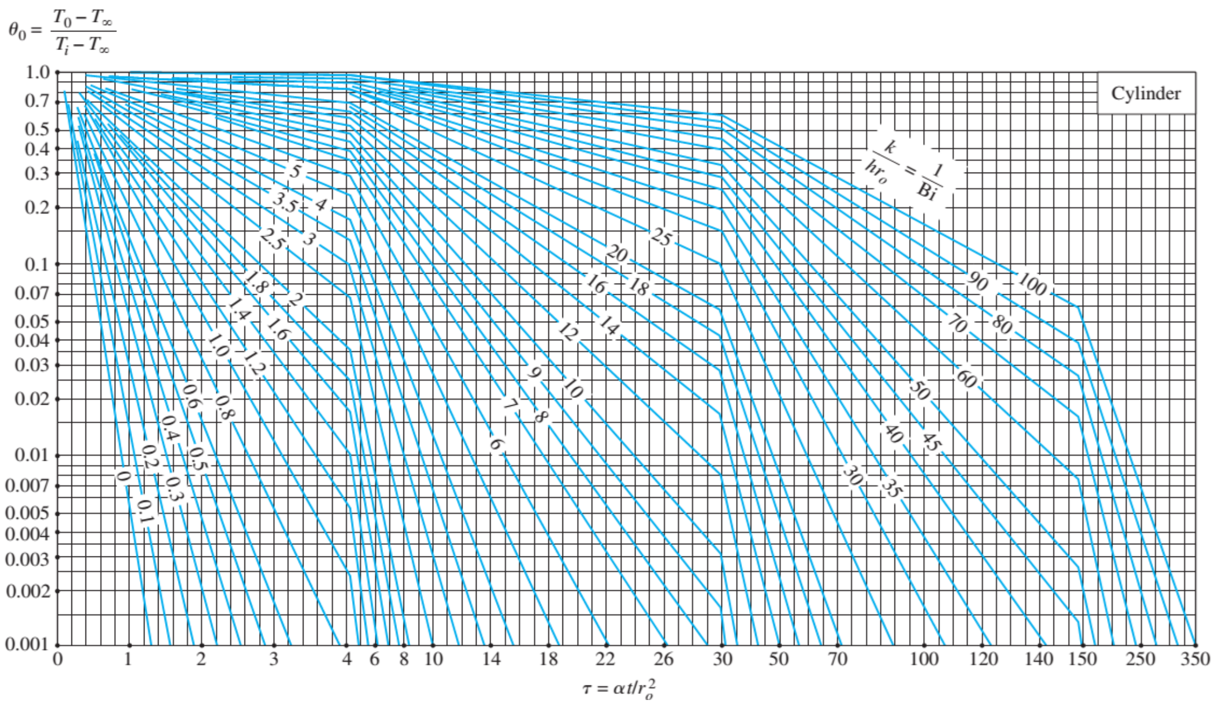


Figure 4.12: Centerline temperature for an infinite cylinder .

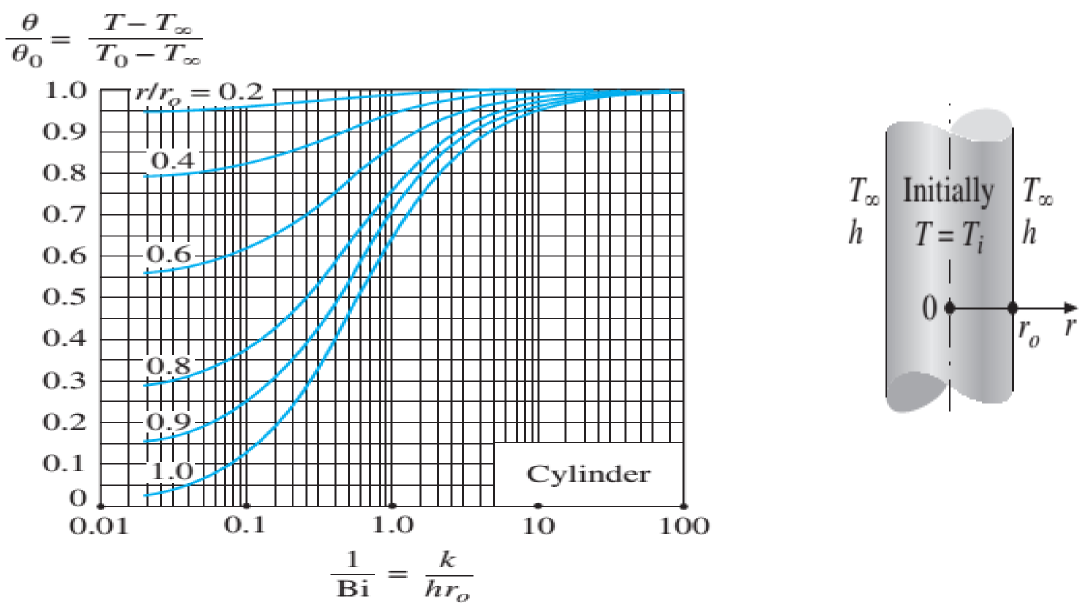


Figure 4.13: Distribution temperature for an infinite cylinder

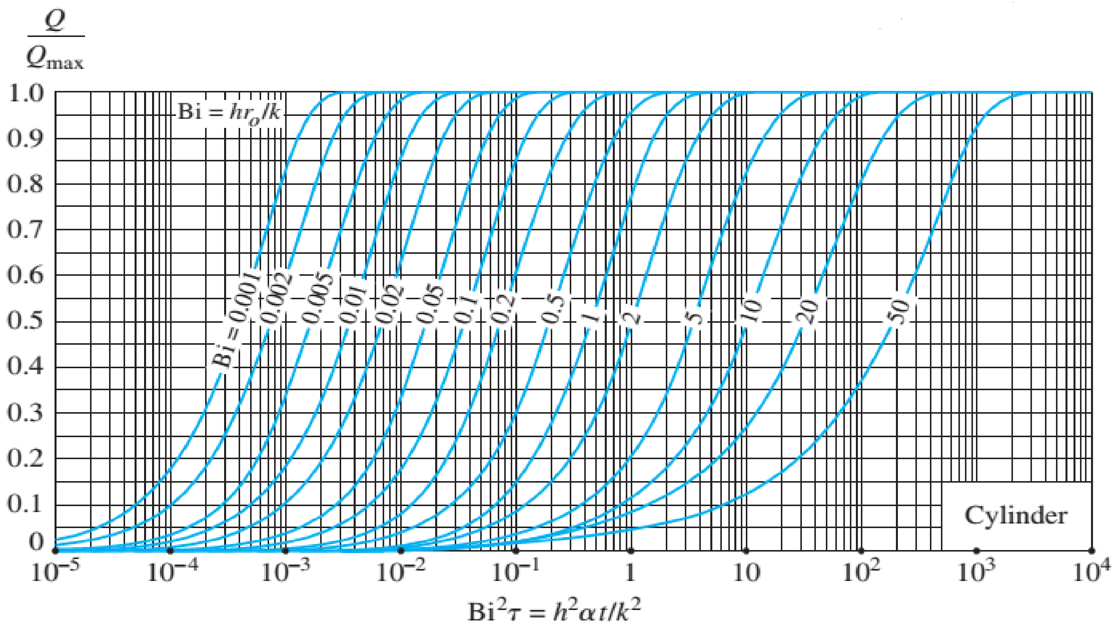


Figure 4.14: Dimensionless heat loss Q/Q_0 of a cylinder.

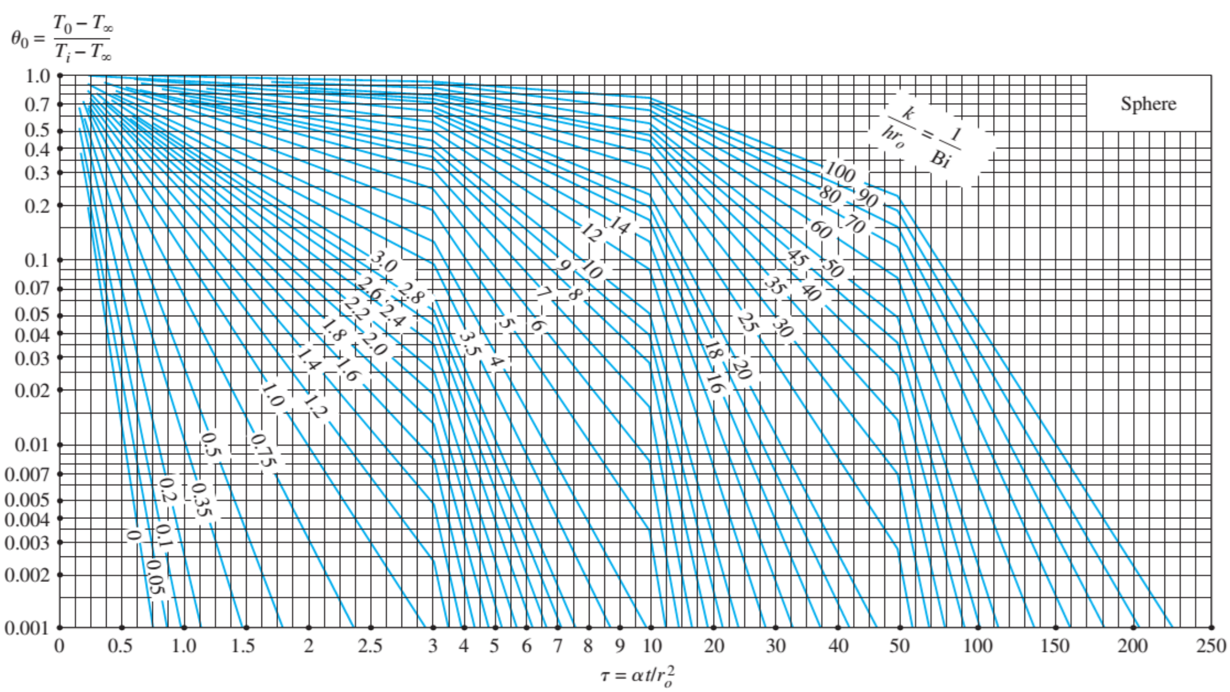


Figure 4.15: Center temperature in a sphere.

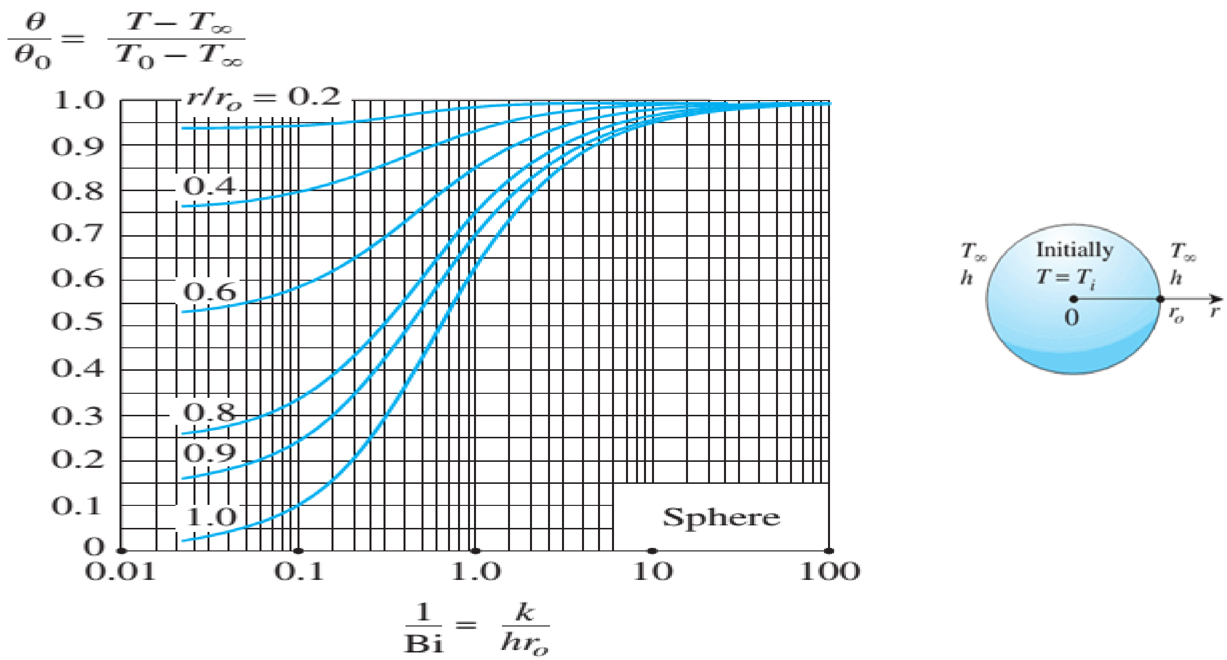


Figure 4.16: Distribution temperature in a sphere .

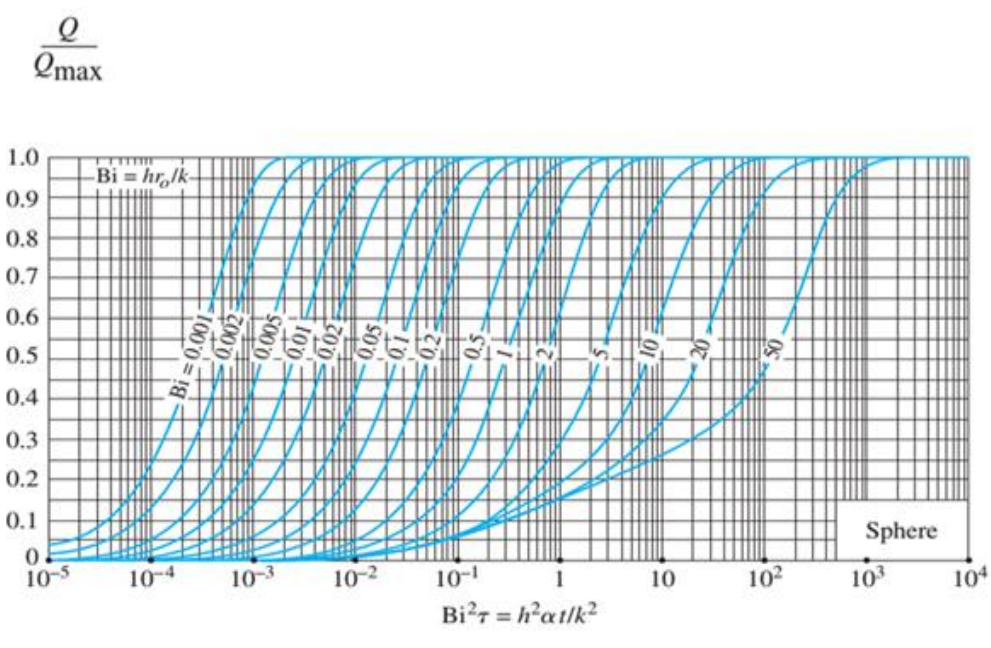


Figure 4.17: Dimensionless heat loss Q/Q_0 of a sphere

4.5 SOLVING THE 1D TRANSIENT HEAT EQUATION USING THE LAPLACE TRANSFORM METHOD

The Laplace Transform method is a powerful tool for solving linear differential equations, especially those modeling transient phenomena such as heat transfer. It converts a differential equation into an algebraic equation in the Laplace domain (the 's' domain), which is often much easier to handle.

The 1D transient heat equation

Consider a semi-infinite wall ($0 \le x < \infty$) initially at a uniform temperature T_i. At time t=0, the surface at x=0 is suddenly raised and maintained at a temperature T_0.

The boundary conditions are:

1. Initial condition: $T(x,0) = T_i$ for $x \ge 0$
2. Boundary condition 1 : $T(0, t) = T_0$ for $t > 0$
3. Boundary condition 2 : $T(\infty, t) = T_i$ for $t > 0$

The general 1D heat equation for a homogeneous, isotropic material without heat generation is :

$$\frac{\partial^2 T}{\partial x^2} - \frac{1}{\alpha}\frac{\partial T}{\partial t} = 0 \tag{4.49}$$

Where: T(x,t) is the temperature as a function of position x and time t.

The Laplace transform of a function T(x, t) is defined as :

$$\overline{T}(x,s) = \int_0^\infty T(x,t)e^{-st}dt \tag{4.50}$$

We apply the Laplace transform with respect to time (t) to the heat equation. We denote : $\overline{T}(x,s) = L\{T(x,t)\}$

$$\int_0^\infty e^{-st}\frac{\partial^2 T}{\partial x^2}dt - \frac{1}{\alpha}\int_0^\infty e^{-st}\frac{\partial T}{\partial t}dt = 0 \tag{4.51}$$

Integration by parts gives :

$$\frac{d^2}{dx^2}\int_0^\infty e^{-st}T(x,t)\,dt - \frac{1}{\alpha}\left[e^{-st}T(x,t)\Big|_{t=0}^{t=\infty} + s\int_0^\infty e^{-st}T(x,t)\,dt\right] = 0 \tag{4.52}$$

$$\frac{d^2\overline{T}}{dx^2} - \frac{1}{\alpha}\left[0 - T(x,0)\right] - \frac{s}{\alpha}\overline{T} = 0 \tag{4.53}$$

Hence

$$\frac{d^2\overline{T}}{dx^2} - \frac{s}{\alpha}\overline{T} = \frac{-T_i}{\alpha} \tag{4.54}$$

This is now an ordinary differential equation (ODE) in x in the Laplace domain

The ODE is a second-order linear equation. The general solution to the homogeneous equation $\frac{d^2\overline{T}}{dx^2} - \frac{s}{\alpha}\overline{T} = 0$ is of the form:

$$\begin{aligned}&\overline{T}(x,s) = A\exp(-kx) + B\exp(kx)\\ &Where:\ k^2 = \frac{s}{\alpha}\end{aligned} \tag{4.55}$$

A particular solution to the complete equation is a constant:

$$\overline{T_p} = \frac{T_i}{s} \tag{4.56}$$

So, the general solution of the ODE is:

$$\overline{T}(x,s) = A\exp(-kx) + B\exp(kx) + \frac{T_i}{s} \tag{4.57}$$

The constants A and B are determined from the boundary conditions:

For x=0 , We have $\overline{T}(0,s) = A + B + \frac{T_i}{s} = \frac{T_0}{s}$

For x $\longrightarrow$ ∞ , we have $\overline{T}(\infty,s) = B + \frac{T_i}{s} = \frac{T_i}{s} \Rightarrow B = 0$

So, $A = \frac{T_0 - T_i}{s}$ .Therefore, $\overline{T}(x,s) = \frac{T_0 - T_i}{s}\exp(-kx) + \frac{T_i}{s}$

Applying the inverse Laplace transform from a given table :

$f(t)$	1	t	t^n	e^{at}	$\cos\ (\omega\ t)$	$sin\ (\omega\ t)$	$erfc\left(\frac{x}{2\sqrt{\alpha t}}\right)$
$F(s)$	$\frac{1}{s}$	$\frac{1}{s^2}$	$\frac{n!}{s^{n+1}}$	$\frac{1}{s-a}$	$\frac{s}{s^2+\omega^2}$	$\frac{\omega}{s^2+\omega^2}$	$\frac{e^{-k\,x}}{s}\ \left(k^2 = \frac{s}{a}\right)$

We obtain:

$$T = (T_0 - T_i)\,erfc\left[\frac{x}{\sqrt{4\alpha t}}\right] + T_i$$

EXERCISES

EXERCISE 4.1

A long cylindrical shaft composed of stainless steel is initially at a uniform temperature T_i of 400 °C. The shaft is allowed to cool slowly in an ambient environment. The thermal properties of the stainless steel at room temperature are:

$\rho = 7900\left[kg/m^3\right], Cp = 477\left[J/kg.K\right], \lambda = 14,9\left[W/\left(m.K\right)\right]$ and $\alpha = 3,95\left[m^2/s\right]$

Determine the following parameters at a specified time

1. The centerline temperature of the cylinder.
2. The total heat transfer per unit length from the cylinder up to that time.

Solution:

Analysis First the Biot number is calculated to be

$$Bi = \frac{h.r_0}{\lambda} = \frac{60\left(0,175\right)}{14,9} = 0,705$$

The constants λ_1 and A corresponding to this Biot number are, from Table 4.2

$$\lambda_1 = 1,0935 \text{ and } A_1 = 1,1558$$

The Fourier number is $\tau = \frac{\alpha t}{{L_c}^2} = \frac{\left(3,95\times10^{-6}\right)\times\left(20\times60\right)}{\left(0,175\right)^2} = 0,1548$

which is very close to the value of 0.2. Therefore, the one-term approximate solution (or the transient temperature charts) can still be used, with the understanding that the error involved will be a little more than 2 percent. Then the temperature at the center of the shaft becomes

$$\theta_{0,\text{cyl}} = \frac{T_0 - T_\infty}{T_i - T_\infty} = A_1 \, e^{-\lambda_1^2 \tau} = \left(1{,}1558\right) e^{-\left(1{,}0935\right)^2 \left(0{,}1548\right)} = 0{,}9605$$

$$\frac{T_0 - 150}{400 - 150} = 0{,}9605 \Rightarrow T_0 = 390°C$$

The maximum heat can be transferred from the cylinder per meter of its length is

$$m = \rho V = \rho \pi r_0^{\,2} L = 7900\pi\left(0{,}175\right)^2.1 = 760{,}1\left[kg\right]$$

$$Q_{\max} = mC_p\left(T_\infty - T_i\right) = 760{,}1\times 477\times\left(400-150\right) = 90638\left[J\right]$$

Once the constant J_1= 0.4689 is determined from Table 4.3 corresponding tothe constant λ_1=1.0935, the actual heat transfer becomes

$$\left(\frac{Q}{Q_{\max}}\right)_{\text{cyl}} = 1 - 2\left(\frac{T_0 - T_\infty}{T_i - T_\infty}\right)\frac{J_1\left(\lambda_1\right)}{\lambda_1} = 1 - 2\left(\frac{390-150}{400-150}\right)\frac{0{,}4689}{1{,}0935} = 0{,}177$$

$$Q = 0{,}177\times 90638 = 16015\left[J\right]$$

<u>EXERCISE 4.2</u>

A steel firewall panel, 5 cm thick, is initially at a uniform temperature of 25 °C. The exterior surface is suddenly exposed to a temperature of 250 °C. Estimate the temperature at the center and at the inner surface of the panel after 20 s of exposure to this temperature.

The thermal diffusivity of the panel is 0,97. 10^{-5} (m^2/s).

Solution

To determine if the panel can be approximated by a semi-infinite solid, we calculate the Fourier number: $\tau = \frac{\alpha t}{L^2} = \frac{0{,}97\times 10^{-5}\times 20}{\left(0{,}05\right)^2} \simeq 0{,}0776$

Since τ < 0.1, the approximation should be acceptable. Thus, using equation (4.13) with x = 0.025 for the temperature at the center

$$\frac{T-T_s}{T_0-T_s}=erf\left(\frac{x}{2\sqrt{\alpha t}}\right)$$
$$\frac{T-250}{25-250}=erf\left(\frac{0.025}{2\sqrt{0,97\times10^{-5}\times20}}\right)=erf\left(0,8974\right)$$
$$\frac{T-250}{-225}=erf\left(0,8974\right)$$
$$T\simeq70.7°C$$

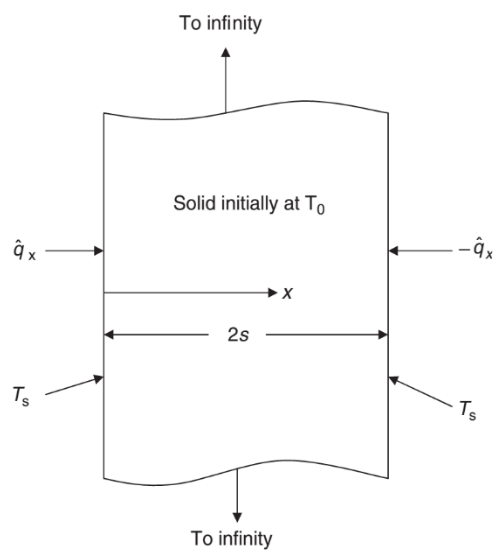


Fig. Infinite solid of finite thickness.

For the interior surface, x = 0.05

$$\frac{T-250}{-225}=erf\left(\frac{0.05}{2\sqrt{0,97\times10^{-5}\times20}}\right)=erf\left(1,795\right)$$
$$=0,9891$$
$$T\simeq27,5°C$$

Thus, the temperature of the interior surface has not changed greatly from its initial value of 25 ◦C, and treating the panel as a semi-infinite solid is therefore a reasonable approximation.

EXERCISE 4.3

A solid spherical,20cm in diameter, has an initial temperature of 400 °C. It is suddenly exposed to air at an ambient temperature T_∞ of 25°C. The convection heat transfer coefficient (h) between the ball and the air is 80 (W/m^2. K)

1. Calculate the time required for the sphere's temperature to decrease to 85°C.
2. Determine the initial rate of cooling at the moment of exposure.
3. Calculate he heat transfer rate from the sphere after 60 seconds.

We give: $\rho = 7800\left[kg/m^3\right], C_p = 450\left[J/kg.K\right], \lambda = 40\left[W/\left(m.K\right)\right]$

Solution

Characteristic length $L_c = \dfrac{V}{A} = \dfrac{R}{3} = 0.00333\left[m\right]$

$$Bi = \frac{h.L_c}{\lambda} = \frac{80\left(0.00333\right)}{40} = 0.00666$$

Since Biot number is <0.1, lamped system analysis can be applied

$$\frac{T - T_\infty}{T_i - T_\infty} = \exp\left(\frac{-h.t}{\rho C_p.L_c}\right)$$

$$\frac{85 - 25}{400 - 25} = \exp\left(\frac{-80.t}{7800\left(450\right)000333}\right)$$

$$t = \frac{7800\left(450\right)0,00333}{80} Ln\left(\frac{375}{60}\right)$$

Therefore $t = 5981.5[s] = 1,66[h]$

Now for initial rate of cooling, we have

$$-mC_p \frac{\partial T}{\partial t} = h.A.(T_i - T_\infty)$$

Or $\dfrac{\partial T}{\partial t} = \dfrac{h.A.(T_i - T_\infty)}{(\rho V)C_p}$

$$\frac{\partial T}{\partial t} = \frac{80}{7800 \times 450 \times 0.00333_p}(400 - 25) = 2,567[°C/s]$$

Instantaneous heat transfer rate is

$$\phi_{in} = h.A.(T - T_\infty)$$

But $(T - T_\infty) = (T_i - T_\infty)\exp\left[-\dfrac{h.t}{\rho C_p.L_c}\right]$

$$\phi_{in} = h.A.(T_i - T_\infty)\exp\left[-\frac{h.t}{\rho C_p.L_c}\right]$$

Here t=60s

$$\phi_{in} = 80.\left(4\pi(0,01)^2\right).(400 - 25)\exp\left[-\frac{80 \times 60}{7800 \times 450 \times 0.00333}\right]$$
$$= 25[W]$$

The total energy transferred during first minute is

$$Q = \rho V.C_p\left(T_i - T_\infty\right)\left[1-\exp\left(-\frac{h.t}{\rho C_p.L_c}\right)\right]$$

$$Q = 7800\left(\frac{4}{3}\pi\left(0,01\right)^3\right)\times 450\times\left(400-25\right)\left[1-\exp\left(-\frac{80\times 60}{7800\times 450\times 0.00333}\right)\right]$$
$$=1856\left[J\right]$$

EXERCISE 4.4

A solid cubical of aluminum with sides measuring 10 mm is initially at a uniform temperature T_i of 50°C. It is placed directly into a flame environment at a temperature T_∞ of 800°C. The convective heat transfer coefficient h between the flame and the aluminum is 190 W/(m^2.K). The desired final temperature T_f of the cube is 300

1°/ Calculate the total time needed for achieving the final temperature of the cube, the value of 300°C.

Solution

For Aluminum: $\rho = 2719\left[kg/m^3\right], C_p = 871\left[J/kg.K\right], \lambda = 215\left[W/\left(m.K\right)\right]$

The characteristic dimension for a cube with side Lis:

$$L_c = \frac{V}{s} = \frac{L^3}{6L^2} = \frac{L}{6}$$

Boot's number is $Bi = \frac{h.L_c}{\lambda} = \frac{190\times 1\times 10^{-2}}{6\times 215} = 0,001473 < 0,1$

As the Biot number is less than 0.1, the lumped heat capacity analysis can be used, which

Gives $\dfrac{T-T_\infty}{T_i-T_\infty}=\exp\left[-\left(\dfrac{hA}{\rho.C_p.V}\right)\tau\right]$

Inserting the given data, we have

$$\frac{300-800}{50-800}=\exp\left[-\left(\frac{190\times 6\times 10^2\times 1000}{871\times 2719\times 1000}\right)\tau\right]$$

Therefore: $\tau=8,41[s]$

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